Math 2390 Homework 7

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1 Short answer

1. For $x, y \in \mathbb{R}$, let $x \sim y$ if $x + y$ is a rational number.
   (a) Is $\sim$ reflexive? If not, find a counterexample.
   (b) Is $\sim$ symmetric? If not, find a counterexample.
   (c) Is $\sim$ transitive? If not, find a counterexample.
2. A relation $\asymp$ is defined on the set $\{1, 2, \ldots, 15\}$ by the following three rules:
   - If x and y are both odd, then $x \asymp y$.
   - If one of x, y is odd and the other is even, then $x \not\asymp y$.
   - If x and y are both even, then $x \asymp y$ if and only if $\frac{x}{2} \asymp \frac{y}{2}$.

   This is an equivalence relation; find its equivalence classes.

2 Proof

3. *This is the first time you're seeing this problem, but I'm not asking you to create your own proof from scratch, just to rewrite this one.*

   Take the following proof by minimum counterexample, and rewrite it as a proof by (strong) induction.

   Definition. The sequence of Fibonacci numbers is defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

   Theorem. Two consecutive Fibonacci numbers are never both even.

   Proof. Suppose the theorem is false, and let n be the least natural number such that F_n and F_{n+1} are both even. Because neither F_1 nor F_2 is even, we know that $n \neq 1$, so $n \geq 2$. Therefore $F_{n+1} = F_n + F_{n-1}$, so $F_{n-1} = F_{n+1} - F_n$.

   Since F_{n+1} and F_n are both even, there are $a, b \in \mathbb{Z}$ such that $F_{n+1} = 2a$ and $F_n = 2b$. But then $F_{n-1} = F_{n+1} - F_n = 2a - 2b = 2(a - b)$, so F_{n-1} is also even. Therefore n is *not* the least natural number such that F_n and F_{n+1} are both even: we could have used F_{n-1} instead.

   This is a contradiction, so our assumption must have been false, and the theorem is true. □

4. *You have already written a rough draft of this proof; now, write a final draft.*

   Let a_n be the sum $2^{0^2} + 2^{1^2} + 2^{2^2} + \cdots + 2^{n^2}$. For example, $a_4 = 2^0 + 2^1 + 2^4 + 2^9 + 2^{16}$.

   Prove by induction on n that for all $n \in \mathbb{N}$, a_n is odd.

5. *For this problem, write a rough draft of an actual proof. I will give you feedback, and you will write a final draft for Homework 8.*

   Let x_n be a sequence that starts at $x_1 = 4$, and continues by the rule $x_{n+1} = \frac{2}{x_n} + \frac{x_n}{2}$ for all $n \in \mathbb{N}$.

   Prove that that the sequence is always decreasing and always stays greater than 2.