Calculus IV Homework 1

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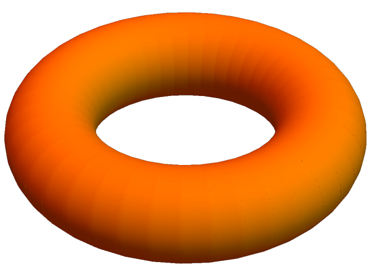
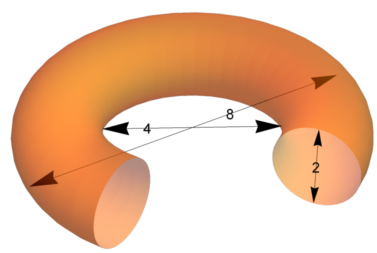
due Friday, January 20, 2023

1 Exercises

1. Given an equation in cylindrical and/or spherical coordinates, describe its graph—either in words, or with a sketch.
   (a) $0 \le \theta \le \frac{\pi}{2}$ and $z = \sqrt{1 - r^2}$ (cylindrical coordinates)
   (b) $0 \le \phi \le \frac{\pi}{3}$ and $0 \le \rho \le \frac{1}{2\cos\phi}$ (spherical coordinates)
   (c) $0 \le z \le 1$ and $r = z^2$ (cylindrical coordinates)
   (d) $r = \rho$ (a mix of cylindrical and spherical coordinates)

2. Consider a donut-shaped region like the one below. The dimensions are given in the second diagram: the hole has diameter 4, a cross-section of the donut has diameter 2, and the diameter of the donut from edge to edge is 8.





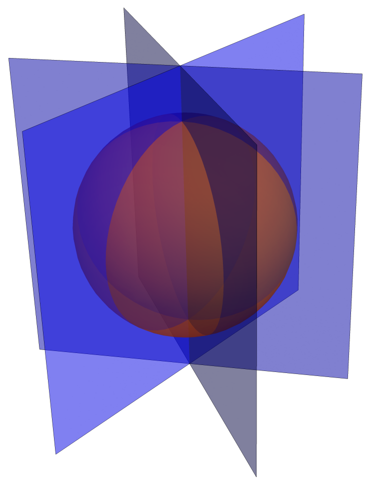
   Use either cylindrical or spherical coordinates, whichever you think is best, to describe this region.

3. Evaluate
$$\int_{-1}^{1}\int_{-3}^{3}\int_{y^2-1}^{y^2+1} (x + y + z)\, dx\, dy\, dz$$
   by applying the transformation $u = x - y^2$, $v = \frac{y}{3}$, $w = \frac{z+1}{2}$ and integrating over an appropriate region in uvw-space.

2 Harder problems

(These problems might be more challenging, but you will benefit from trying them. Even if you don't fully succeed, I encourage you to write things like "I don't know how to do this next step, but if I could do it, I would finish the problem by doing thus-and-such" or "I must have made a mistake at some point, because this result doesn't have the whatsit form I expected" and will assign partial credit accordingly.)

4. A solid sphere of radius 1 is sliced into six equal wedges by three vertical planes passing through the z-axis.



   Determine the distance between the center of the sphere and the center of mass of one of the wedges.

5. In probability theory, the theory of substitution in multiple integrals has a different use: not to evaluate integrals, but for its own sake.

   For example, suppose that x and y are two real numbers picked uniformly at random from the interval $[1, 2]$, and we want to know: what is the distribution of the ratio $\frac{x}{y}$?

   To do this, we start with the very boring integral $\int_1^2 \int_1^2 1\, dx\, dy$ which we know is equal to 1. Then, we make a substitution with $u = \frac{x}{y}$ and $v = x$, getting an integral $\iint_{S'} |J(u, v)|\, dv\, du$ where S' is the region in the uv-plane corresponding to our original square in the xy-plane. Finally, we evaluate only the inside integral (with respect to v) to get an integral of the form $\int_a^b f(u)\, du$.

   The function $f(u)$ ranging from a to b will now be the probability density function describing the distribution of $u = \frac{x}{y}$. (If you're not familiar with PDFs, what this means is that a graph of $f(u)$ will tell you which values of u are more and less likely; area under that curve corresponds to probability.)

   Your job in this problem is to go and do that and find the function $f(u)$.