Calculus IV Homework 3

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1 Exercises

1. Let $\mathbf{F} = 2y\,\mathbf{i} + y\,\mathbf{j}$.
   (a) Draw a sketch of $\mathbf{F}$ in the region $-2 \le x \le 2$, $-2 \le y \le 2$.
   (b) Find the flux integral of $\mathbf{F}$ around the boundary of the region bounded by the parabola $y = x^2$ and the line $y = 1$.
2. Determine whether these vector fields are gradient fields, and if they are, find the potential function.
   (a) $\mathbf{F} = z\cos y\,\mathbf{i} - xz\sin y\,\mathbf{j} + ((z+1)e^z + x\cos y)\,\mathbf{k}$.
   (b) $\mathbf{G} = 2xyz\,\mathbf{i} + x^2z\,\mathbf{j} + xy^2\,\mathbf{k}$.
3. Find a potential function for $\mathbf{F} = (y + e^{x-y})\,\mathbf{i} + (x - e^{x-y})\,\mathbf{j}$, and use it to find the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \le t \le \frac{3\pi}{2}$.

2 Harder problems

4. Let $\mathbf{F}$ be an unknown, but conservative, vector field. Define paths
   - C_1, the line segment parametrized by $\mathbf{r}_1(t) = (2t-1, 0)$ for $0 \le t \le 1$.
   - C_2, the quarter-circle parametrized by $\mathbf{r}_2(t) = (\cos t, \sin t)$ for $0 \le t \le \frac{\pi}{2}$.
   - C_3, the parabolic curve parametrized by $\mathbf{r}_3(t) = \left(\frac{-t-t^2}{2}, \frac{t-t^2}{2}\right)$ for $-1 \le t \le 1$.
   - C_4, the cubic curve parametrized by $\mathbf{r}_4(t) = (t, t^3 - t)$ for $-1 \le t \le 1$.

   If $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = 1$, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2$, and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}_3 = 3$, find $\int_{C_4} \mathbf{F} \cdot d\mathbf{r}_4$.
5. The path C parametrized by $\mathbf{r}(t) = (t\cos t, t\sin t, t)$ as $\frac{\pi}{2} \le t \le \pi$ winds around a portion of the cone $z^2 = x^2 + y^2$. You decide to compute the integral

$$\int_C \left(\frac{x\,\mathbf{i} + y\,\mathbf{j} + 0\,\mathbf{k}}{x^2 + y^2 + z^2}\right) \cdot d\mathbf{r}$$

but then you spot a shortcut: the closely related vector field $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{x^2 + y^2 + z^2}$ is conservative, with potential function $\frac{1}{2}\ln(x^2 + y^2 + z^2)$.

(a) Find the integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$. *(This shouldn't actually require taking integrals.)*

(b) Use your answer to (a) to replace the integral $\displaystyle\int_C \left(\frac{x\,\mathbf{i} + y\,\mathbf{j} + 0\,\mathbf{k}}{x^2 + y^2 + z^2} \right) \cdot d\mathbf{r}$ you actually want by a much easier integral, then take that integral.