Calculus IV Homework 5

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1 Exercises

1. Find parameterizations for the following surfaces:

   (a) The portion of a sphere with radius 2 centered at $(0,0,0)$ which has nonnegative x-, y-, and z-coordinates.

   (b) The portion of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.

   (c) The flat diamond (rhombus) with corners at $(1,0,1), (0,1,1), (0,0,2), (1,1,0)$.

2. Write down an integral to find the area of the surface parameterized by

$$\mathbf{r}(u,v) = (u\cos v, u\sin v, u)$$

   where $-1 \le u \le 1$ and $0 \le v \le \pi$. Do not evaluate the integral (but do simplify the integrand as much as possible).

2 Harder problems

3. Green's theorem seems to have two completely separate statements for flux and circulation, but actually they are the same statement, we just have two separate ways to talk about it. This problem will give you some intuition for why that is.

   Let C be the unit circle with the usual counterclockwise parameterization $\mathbf{r}(t) = (\cos t, \sin t)$ as $0 \le t \le 2\pi$. Let

$$I = \int_0^{2\pi} e^{\sin t}(\sin t + \cos t)\, dt.$$

   (a) Find a vector field $\mathbf{F}_1$ such that the circulation of $\mathbf{F}$ around C is given by the integral I.

   (b) Find a vector field $\mathbf{F}_2$ such that the flux of $\mathbf{F}$ across C is given by the integral I.

   (c) Verify that the circulation density of $\mathbf{F}_1$ and the flux density of $\mathbf{F}_2$ are equal. This means that whether we think of I as a circulation integral for $\mathbf{F}_1$ or as a flux integral for $\mathbf{F}_2$, Green's theorem will turn it into the same double integral over the interior of C!

4. Example 3 in section 16.5 of the textbook parameterizes the cylinder

$$x^2 + (y-3)^2 = 9, \quad 0 \le z \le 5$$

by $\mathbf{r}(\theta, z) = (3\sin 2\theta, 6\sin^2\theta, z)$ as $0 \le \theta \le \pi$, $0 \le z \le 5$. There is another way to do it:

(a) Find a parameterization of the cylinder

$$x^2 + y^2 = 9, \quad 0 \le z \le 5$$

inspired by cylindrical coordinates. (You will want to have $0 \le \theta \le 2\pi$.)

(b) We want a cylinder centered at $x = 0$ and $y = 3$ rather than $x = 0$ and $y = 0$. Apply a translation to your answer to part (a) to get a different parameterization of the cylinder in Example 3 of the textbook.

(c) Show that if, instead of having $0 \le \theta \le 2\pi$, we replace θ by 2θ in the parameterization and have θ range from 0 to π, we get an identical parameterization to the textbook's.