Calculus IV Homework 7

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due Friday, April 14, 2023

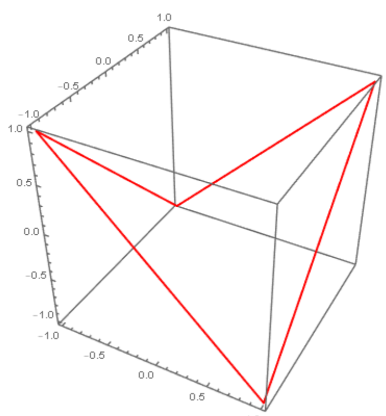
1 Exercises

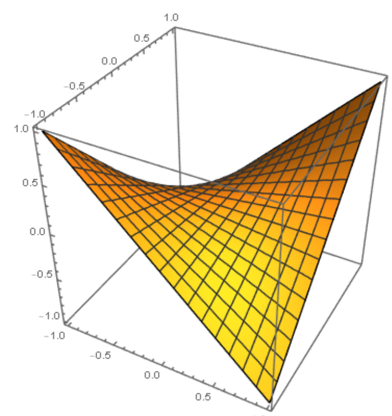
1. Find the curl $\nabla \times \mathbf{F}$ of each vector field $\mathbf{F}$:
   (a) $\mathbf{F} = e^{e^{e^x}}\,\mathbf{i} + \sqrt{y + \sqrt{y + \sqrt{y}}}\,\mathbf{j} + \cos^7 z\,\mathbf{k}$.
   (b) $\mathbf{F} = y\ln(x+y+z)\,\mathbf{i} + x\ln(x+y+z)\,\mathbf{j} + z\ln(x+y+z)\,\mathbf{k}$.
   (c) $\mathbf{F} = (x+2z)\,\mathbf{i} + (3x+4y)\,\mathbf{j} + (5y+6z)\,\mathbf{k}$
   (d) $\mathbf{F} = z^3\,\mathbf{i} + (x - z^3)\,\mathbf{j}$.

2 Harder problems

(I feel like none of the problems involving Stokes' theorem can be called "exercises", given how much there is going on here, so they're all in the "harder problems" section.)

2. Let C be the closed curve made up of four straight line segments: from $(1,1,1)$ to $(-1,1,-1)$, from $(-1,1,-1)$ to $(-1,-1,1)$, from $(-1,-1,1)$ to $(1,-1,-1)$, and from $(1,-1,-1)$ back to $(1,1,1)$. This is shown in the diagram on the left.





Fun fact: this weird piecewise linear curve C actually lies entirely on the graph of $z = xy$, shown in the diagram on the right.

(a) Use this fun fact, and Stokes' theorem, to find the circulation integral

$$\int_C (z^2\,\mathbf{i} + x^2\,\mathbf{j} + y^2\,\mathbf{k}) \cdot \mathbf{T}\,ds$$

(in a counterclockwise direction as seen from above) by first turning it into a surface integral.

(b) For comparison (and to review), find the circulation integral directly as a line integral, but just for **one** of the line segments (your choice).

(The line segments are all basically the same, so doing more would just be busy work, though you can do it if you like.)

3. Let S be the surface which consists of the portion of the cylinder $x^2 + y^2 = 1$ with $0 \le z \le 1$, and let $\mathbf{F} = 2yz\,\mathbf{i} + 3x\,\mathbf{j} - z^2\,\mathbf{k}$.

(a) Find a parameterization $\mathbf{r}(u, v)$ of S.

(b) Find the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma$$

using your parameterization, with a normal vector pointing outward (away from the z-axis).

(c) Using Stokes' theorem, the integral in (b) can be expressed in terms of *two* circulation integrals of $\mathbf{F}$ around certain curves in $\mathbb{R}^3$. How?

4. (a) Find a surface S whose boundary is the curve C parameterized by

$$\mathbf{r}(t) = (\cos t, \sin t, \cos(2t) + \sin(2t)).$$

(Hint: look up the double-angle formulas if you don't remember them.)

(b) Use Stokes' theorem to turn the circulation integral of $\mathbf{F} = z^3\,\mathbf{i} + (x - z^3)\,\mathbf{j}$ around C into a flux integral across S, and then evaluate that integral.