Differential forms for Calculus IV

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1 What are differential forms?

Differential forms are meant to represent all the (sufficiently nice) things we can evaluate or integrate over some (sufficiently nice) region in $\mathbb{R}^n$. I will stick to $\mathbb{R}^3$ in these notes, but everything generalizes to higher dimensions without effort if you want to try.

In $\mathbb{R}^3$, differential forms come in four types:

- The 0-**forms**, which are smooth (infinitely differentiable) functions $f\colon \mathbb{R}^3 \to \mathbb{R}$.
- The 1-**forms**: expressions of the form $M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$ where M, N, and P are 0-forms.

  For the most part, we will not need to think too hard about what $\mathrm{d}x$ is, just as we have not thought too hard about it too hard in the past. "An infinitesimal change in the positive x-direction" is probably good enough to go on.
- The 2-**forms**: expressions of the form $U\,\mathrm{d}y\,\mathrm{d}z + V\,\mathrm{d}z\,\mathrm{d}x + W\,\mathrm{d}x\,\mathrm{d}y$ where U, V, W are 0-forms.

  Here, a similarly not-quite-rigorous way of thinking about $\mathrm{d}x\,\mathrm{d}y$ is as "an infinitesimal square parallel to the xy-plane, oriented with a normal vector in the positive z-direction". The "oriented" bit is important; $\mathrm{d}y\,\mathrm{d}x$ will be oriented in the opposite way.
- The 3-**forms**: expressions of the form $g\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where g is a 0-form.

In practice, every time we have a scalar field in this class, we can think of it as either a 0-form or a 3-form; every time we have a vector field, we can think of it as either a 1-form or a 2-form.

Which one to pick? Sometimes there is a right answer to that—rotation vectors should be represented by 2-forms, not 1-forms. Since we will not go too far into the depths of working with differential forms, we will generally just pick whichever one is convenient.

2 The exterior product

The first useful thing we get out of thinking of our scalar fields and vector fields as differential forms is the exterior product. This is an operation that lets us multiply two differential forms to get another. It will generalize operations like dot product and cross product that you are familiar with from vector calculus (and in higher dimensions, it can do other, weirder things).

(A note on notation: when people seriously work with differential forms, they often use the symbol $\wedge$ for exterior product, writing $(x^2\,\mathrm{d}x + 2xy\,\mathrm{d}y) \wedge (xyz\,\mathrm{d}z)$ for the exterior product of $x^2\,\mathrm{d}x + 2xy\,\mathrm{d}y$ with $xyz\,\mathrm{d}z$. We will not do this, because there will be no other products to be confused with, so it will just make everything scarier.)

Defining the exterior product is a lot like teaching you how to multiply complex numbers when you already know how to multiply real numbers. By distributing, we can simplify a product $(a + bi)(c + di)$ to get $ac + (ad + bc)i + bdi^2$. All we need to be told is that $i^2 = -1$, and we can simplify this product to its final form $(ac - bd) + (ad + bc)i$.

Similarly, in order to multiply differential forms, we just need to know how to multiply the differential elements $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$. Part of the answer to how to do this is this: **just leave the product alone!** The exterior product of $\mathrm{d}x$ and $\mathrm{d}y$ is just $\mathrm{d}x\,\mathrm{d}y$. Differential elements **commute with 0-forms**, so if we multiply $x\,\mathrm{d}x$ by $x\,\mathrm{d}y$, we can write the product as $x^2\,\mathrm{d}x\,\mathrm{d}y$: the product of two 1-forms is a 2-form.

But if you multiply out an expression like $(x\,\mathrm{d}x - y\,\mathrm{d}y)(x\,\mathrm{d}x + y\,\mathrm{d}y)$, what you get is

$$x^2\,\mathrm{d}x\,\mathrm{d}x + xy\,\mathrm{d}x\,\mathrm{d}y - xy\,\mathrm{d}y\,\mathrm{d}x - y^2\,\mathrm{d}y\,\mathrm{d}y.$$

This looks a lot *like* a 2-form, but only one term of it is part of how we described 2-forms: the $xy\,\mathrm{d}x\,\mathrm{d}y$ term.

To get the rest of the expression there, we have two rules for multiplying differential elements:

1. For any two variables u and v, we have $\mathrm{d}u\,\mathrm{d}v = -\mathrm{d}v\,\mathrm{d}u$.

   The intuition here is that $\mathrm{d}u\,\mathrm{d}v$ is an infinitesimal *oriented* square, and $\mathrm{d}v\,\mathrm{d}u$ is the same square oriented with the opposite orientation.

2. For any variable u, we have $\mathrm{d}u\,\mathrm{d}u = 0$.

   We can deduce this from the previous rule: we should be able to swap the two differentials, so we should have $\mathrm{d}u\,\mathrm{d}u = -\mathrm{d}u\,\mathrm{d}u$, and only 0 has this property.

   The other intuition is that a product $\mathrm{d}u\,\mathrm{d}v$ is supposed to represent a square with sides parallel to $\mathrm{d}u$ and $\mathrm{d}v$; more generally, if these vectors were not perpendicular, we would have a parallelogram. But a parallelogram with adjacent sides both parallel to $\mathrm{d}u$ has area 0.

These rules let us simplify the product in the example above:

$$\begin{aligned}(x\,\mathrm{d}x - y\,\mathrm{d}y)(x\,\mathrm{d}x + y\,\mathrm{d}y) &= x^2\,\mathrm{d}x\,\mathrm{d}x + xy\,\mathrm{d}x\,\mathrm{d}y - xy\,\mathrm{d}y\,\mathrm{d}x - y^2\,\mathrm{d}y\,\mathrm{d}y \\ &= x^2(0) + xy\,\mathrm{d}x\,\mathrm{d}y - xy(-\mathrm{d}x\,\mathrm{d}y) - y^2(0) \\ &= xy\,\mathrm{d}x\,\mathrm{d}y + xy\,\mathrm{d}x\,\mathrm{d}y \\ &= 2xy\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

So that's how to compute the exterior product; but what does it do? The answer is: it does everything!

In general, multiplying a k-form by an l-form will give a $(k+l)$-form. If $k+l$ is bigger than the dimension we're working in, we'll always get 0, due to the rule that $\mathrm{d}u\,\mathrm{d}u = 0$. Also, multiplying by a 0-form is the same as multiplying by a scalar. So the two interesting cases to look at in $\mathbb{R}^3$ are: multiplying two 1-forms, and multiplying a 1-form with a 2-form.

If we multiply two 1-forms $(M_1\,\mathrm{d}x + N_1\,\mathrm{d}y + P_1\,\mathrm{d}z)(M_2\,\mathrm{d}x + N_2\,\mathrm{d}y + P_2\,\mathrm{d}z)$, we get a 9-term product:

$$\begin{array}{rrr} M_1M_2\,\mathrm{d}x\,\mathrm{d}x & +M_1N_2\,\mathrm{d}x\,\mathrm{d}y & +M_1P_2\,\mathrm{d}x\,\mathrm{d}z \\ +N_1M_2\,\mathrm{d}y\,\mathrm{d}x & +N_1N_2\,\mathrm{d}y\,\mathrm{d}y & +N_1P_2\,\mathrm{d}y\,\mathrm{d}z \\ +P_1M_2\,\mathrm{d}z\,\mathrm{d}x & +P_1N_2\,\mathrm{d}z\,\mathrm{d}y & +P_1P_2\,\mathrm{d}z\,\mathrm{d}z \end{array}$$

The diagonal terms cancel. For the off-diagonal terms, we reorder the differentials to match $U\,\mathrm{d}y\,\mathrm{d}z + V\,\mathrm{d}z\,\mathrm{d}x + W\,\mathrm{d}x\,\mathrm{d}y$: our model for a 2-form. We get

$$(N_1P_2 - P_1N_2)\,\mathrm{d}y\,\mathrm{d}z + (P_1M_2 - M_1P_2)\,\mathrm{d}z\,\mathrm{d}x + (M_1N_2 - N_1M_2)\,\mathrm{d}x\,\mathrm{d}y.$$

What is this? This is the cross product of the vectors (M_1, N_1, P_1) and (M_2, N_2, P_2). So that's our first observation: **the exterior product of two 1-forms is their cross product, as a 2-form.**

What happens if we multiply a 1-form $M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$ by a 2-form $U\,\mathrm{d}y\,\mathrm{d}z + V\,\mathrm{d}z\,\mathrm{d}x + W\,\mathrm{d}x\,\mathrm{d}y$? Before simplifying anything, we'll get another 9-term product:

$$\begin{array}{rrr} MU\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z & +MV\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}x & +MW\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}y \\ +NU\,\mathrm{d}y\,\mathrm{d}y\,\mathrm{d}z & +NV\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x & +NW\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}y \\ +PU\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}z & +PV\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}z & +PW\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \end{array}$$

In this 9-term product, every off-diagonal term contains a repeated variable, so it simplifies to 0. The first term, $MU\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, is already written like our model 3-form. To combine the middle term with it, we rewrite $NV\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$ first as $-NV\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$ and then as $NV\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. (Every time we swap two differential elements, the sign changes, but we do two swaps.) For the last term, we get $PW\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = -PW\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = PW\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. The final answer is

$$(MU + NV + PW)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

In other words: **the exterior product of a 1-form with a 2-form is their dot product, as a 3-form**.

3 The exterior derivative

It's kind of nice that we get cross products and dot products as special cases of the same operation. But one of the biggest advantages of working with differential forms is that we get a single operation—the exterior derivative—that generalizes gradient, curl, and divergence.

This operation is called the exterior derivative. It turns 0-forms into 1-forms, 1-forms into 2-forms, and 2-forms into 3-forms. Here is how it is defined:

- The exterior derivative of a 0-form f is

  $$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z.$$

  We will use this to define all the other exterior derivatives.

- The exterior derivative of a 1-form is

$$\mathrm{d}(M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z) = (\mathrm{d}M)\,\mathrm{d}x + (\mathrm{d}N)\,\mathrm{d}y + (\mathrm{d}P)\,\mathrm{d}z$$

  where $\mathrm{d}M$, $\mathrm{d}N$, and $\mathrm{d}P$ are defined by the rule for 0-forms. This expression has some exterior products in it, and we'll use the rules for those to simplify it.

- The exterior derivative of a 2-form is

$$\mathrm{d}(U\,\mathrm{d}y\,\mathrm{d}z + V\,\mathrm{d}z\,\mathrm{d}x + W\,\mathrm{d}x\,\mathrm{d}y) = (\mathrm{d}U)\,\mathrm{d}y\,\mathrm{d}z + (\mathrm{d}V)\,\mathrm{d}z\,\mathrm{d}x + (\mathrm{d}W)\,\mathrm{d}x\,\mathrm{d}y$$

  where $\mathrm{d}U$, $\mathrm{d}V$, and $\mathrm{d}W$ are defined by the rule for 0-forms. Once again, this expression has some exterior products in it, and we'll use the rules for those to simplify it.

In general, the idea is that when we take the exterior derivative of a 0-form multiplied by some differential elements, we first take the exterior derivative of the 0-form, and then multiply the differential elements on at the end.

The definition of $\mathrm{d}f$ also lets us change coordinates, and work with differential forms in a different basis. For example, we could work with differential forms in cylindrical coordinates, written in terms of $\mathrm{d}r$, $\mathrm{d}z$, and $\mathrm{d}\theta$. We will not do this very much, but some results of that are interesting: for example, $\mathrm{d}\theta$, when written in rectangular coordinates, is

$$\mathrm{d}\theta = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

which is a vector field with a singularity along the z-axis that we've used as a counterexample several times in this class.

Our definition

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

makes the exterior derivative look a lot like the gradient—and it is the gradient, when applied to 0-forms. (The gradient ∇f is a vector field, and the 1-form $\mathrm{d}f$ can be interpreted as a vector field.)

Meanwhile, if we take the exterior derivative of a 1-form, we get its curl—as a 2-form. The calculation here, again, involves a 9-term exterior product with lots of cancellation. So let's just do an example: what is the exterior derivative of $z^3\,\mathrm{d}x + (x-z^3)\,\mathrm{d}y$?

- The $z^3\,\mathrm{d}x$ term turns into $(\frac{\partial}{\partial x}z^3\,\mathrm{d}x + \frac{\partial}{\partial y}z^3\,\mathrm{d}y + \frac{\partial}{\partial z}z^3\,\mathrm{d}z)\,\mathrm{d}x$, or $3z^2\,\mathrm{d}z\,\mathrm{d}x$.
- The $(x-z^3)\,\mathrm{d}y$ term turns into $(\frac{\partial}{\partial x}(x-z^3)\,\mathrm{d}x + \frac{\partial}{\partial y}(x-z^3)\,\mathrm{d}y + \frac{\partial}{\partial z}(x-z^3)\,\mathrm{d}z)\,\mathrm{d}y$, which simplifies to $(\mathrm{d}x - 3z^2\,\mathrm{d}z)\,\mathrm{d}y$ or $3z^2\,\mathrm{d}y\,\mathrm{d}z + \mathrm{d}x\,\mathrm{d}y$.

Altogether, we get $3z^2\,\mathrm{d}y\,\mathrm{d}z + 3z^2\,\mathrm{d}z\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}y$. This should look familiar: for homework, you computed the curl of $\mathbf{F} = z^3\,\mathbf{i} + (x-z^3)\,\mathbf{j}$ to be $3z^2\,\mathbf{i} + 3z^2\,\mathbf{j} + \mathbf{k}$.

The exterior derivative of a 2-form gives a 3-form, and it is also a familiar operation: the divergence.

There are two facts we have learned relating the gradient, curl, and divergence (with no differential forms involved):

- The curl of a gradient field is 0. (We previously called this the component test.)
- The divergence of a curl field is 0.

In the language of differential forms, this says: for any differential form ω, $\mathrm{d}(\mathrm{d}\omega) = 0$. Applying the exterior derivative twice always cancels out to 0. Essentially, this happens because mixed partial derivatives end up canceling.

4 Integrating a differential form

If we have a k-form in n variables, we can integrate it over an oriented k-dimensional subset of $\mathbb{R}^n$. For our purposes it means that we can:

- Integrate a 1-form $M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$ over an oriented curve in $\mathbb{R}^3$.
- Integrate a 2-form $U\,\mathrm{d}y\,\mathrm{d}z + V\,\mathrm{d}z\,\mathrm{d}x + W\,\mathrm{d}x\,\mathrm{d}y$ over an oriented surface in $\mathbb{R}^3$.
- Integrate a 3-form $g\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ over an oriented solid region in $\mathbb{R}^3$.

Here, too, I think that using differential forms demonstrates that the various integrals we've done this semester, which have often seemed somewhat arbitrary, are actually the correct integrals to consider! Circulation and flux integrals all spontaneously appear when we figure out what exactly integrating a differential form should mean.

Let's begin with the integral over a curve.

Compared to the other uses of differential forms, the integral

$$\int_C M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$$

is a much more familiar object. Your textbook even uses it as notation for some line integrals.

The most flexible approach to taking this integral is to parameterize the curve C: a function $\mathbf{r}(t)$ such that as t goes from a to b, the point $\mathbf{r}(t)$ traces out the curve C. We have already discussed such line integrals in this class. The only caveat: for this material, it is convenient to think of the curve C itself as having an orientation. The parameterization $\mathbf{r}$ must be chosen to be consistent with that orientation.

We think of $\mathbf{r}$ as giving a different coordinate system: a 1-dimensional coordinate system that can only be used for points on the curve C. In that 1-dimensional coordinate system, all differential forms can be expressed in terms of the differential element $\mathrm{d}t$. Our native-to-$\mathbb{R}^3$ differential elements $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ must be expressed in terms of $\mathrm{d}t$: as $\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$, $\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$, and $\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t$, respectively. What are $\frac{\mathrm{d}x}{\mathrm{d}t}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$, and $\frac{\mathrm{d}z}{\mathrm{d}t}$? They are components of the derivative $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$.

So we rewrite the integral as

$$\int_a^b M\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t + N\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t + P\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t,$$

turning it into a normal 1-variable integral with respect to t. Importantly, M, N, and P should all be turned into functions of t, by evaluating them at $\mathbf{r}(t)$. The quantity $M\frac{\mathrm{d}x}{\mathrm{d}t} + N\frac{\mathrm{d}y}{\mathrm{d}t} + P\frac{\mathrm{d}z}{\mathrm{d}t}$ is what we're used to writing as $\mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$. We get back the ordinary line integral of a vector field around a curve!

The main difficulty with the integral of a 1-form is not misinterpreting the notation. It is very tempting to look at an integral like

$$\int_C x^3 yz \, \mathrm{d}x$$

and think of taking an *antiderivative* of x^3yz with respect to x, getting some expression like $\frac{1}{4}x^4yz$ to be evaluated... somewhere. It's often not clear where to evaluate it, but more importantly, getting to $\frac{1}{4}x^4yz$ is often a mistake to begin with: it assumes that y and z are constants with respect to x, and usually they are not! In special cases, where C is a line segment aligned in one cardinal direction, we can get somewhere with this approach. Unless you are in that case and *very* certain you know what you are doing, don't attempt it.

Now let's talk about surface integrals of 2-forms: the integral

$$\iint_S U \, \mathrm{d}y \, \mathrm{d}z + V \, \mathrm{d}z \, \mathrm{d}x + W \, \mathrm{d}x \, \mathrm{d}y$$

where S is an oriented surface.

Here, we will also want to choose a parameterization: a function $\mathbf{r}(u, v)$ such that as u and v vary over some domain D, the point $\mathbf{r}(u, v)$ traces out the surface S. Once again, the surface S already has an orientation, and $\mathbf{r}$ must be chosen to be consistent with that orientation.

The parameterization gives a 2-dimensional coordinate system for points on the surface S, and so the differential elements $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ must be expressed in terms of the differential elements $\mathrm{d}u$ and $\mathrm{d}v$ in this 2-dimensional coordinate system. This is done by writing

$$\mathrm{d}x = \frac{\partial x}{\partial u} \, \mathrm{d}u + \frac{\partial x}{\partial v} \, \mathrm{d}v, \quad \mathrm{d}y = \frac{\partial y}{\partial u} \, \mathrm{d}u + \frac{\partial y}{\partial v} \, \mathrm{d}v, \quad \mathrm{d}z = \frac{\partial z}{\partial u} \, \mathrm{d}u + \frac{\partial z}{\partial v} \, \mathrm{d}v.$$

The partial derivatives such as $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \dots$ are the appropriate components of the partial derivatives $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$.

But our differential form is a 2-form, with expressions such as $\mathrm{d}x \, \mathrm{d}y$ in it, which must be simplified by the exterior product. We have

$$\mathrm{d}x \, \mathrm{d}y = \left(\frac{\partial x}{\partial u} \, \mathrm{d}u + \frac{\partial x}{\partial v} \, \mathrm{d}v \right) \left(\frac{\partial y}{\partial u} \, \mathrm{d}u + \frac{\partial y}{\partial v} \, \mathrm{d}v \right) = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v.$$

In non-differential terms, we can recognize this as the $\mathbf{k}$-component of $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$. Expanding out $\mathrm{d}y \, \mathrm{d}z$ and $\mathrm{d}z \, \mathrm{d}x$ in the same way gives the $\mathbf{i}$- and $\mathbf{j}$-components. We end up rewriting our integrand as

$$\iint_D U \left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v + V \left(\frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v + W \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v$$

and this is exactly the expression $\mathbf{F} \cdot (\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v})$ that we usually see in our surface integrals![1]

[1] Without using the exterior product, there is a less fancy way to turn $U \, \mathrm{d}y \, \mathrm{d}z$ into $U \left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v$: the factor we see there is also the Jacobian determinant $J(u, v)$. This makes sense: what we're doing here is really a uv-substitution. We do not take the absolute value of the Jacobian, because we care about the orientation of our surface.

Finally, we can integrate a 3-form:

$$\iiint_R g \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where R is a solid region in $\mathbb{R}^3$. In this case, we don't have to choose a parameterization to make sense of this integral; this is just the ordinary triple integral over R!

However, we *could* choose a parameterization of R; a function $\mathbf{r}(u, v, w)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$. (We usually call this a "transformation" or "change of coordinates"). If we write $\mathrm{d}x = \frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v + \frac{\partial x}{\partial w}\,\mathrm{d}w$, do the same thing for $\mathrm{d}y$ and $\mathrm{d}v$, and then expand out the exterior product $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, what will we get? We'll get $\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$ with a factor in front: a factor we are already familiar with as the Jacobian determinant of the transformation $\mathbf{r}$.

The fact the regions we integrate over are all oriented is important here: it means that to get the correct sign on our integral, the transformation $\mathbf{r}$ must be orientation-preserving. (If applied to an asymmetric shape in uvw-space, it must produce some possibly-distorted copy of that shape in xyz-space, as opposed to a distorted *mirror image* of that shape.)

This is easily addressed. If $\mathbf{r}$ is not orientation-preserving, the Jacobian determinant will be negative. To fix this, we make sure to take the absolute value of the Jacobian determinant.

5 The generalized Stokes' theorem

All this enables us to state a theorem known as the generalized Stokes' theorem. It has a very simple statement:

Let M be a d-dimensional oriented manifold[2] in $\mathbb{R}^n$ with boundary ∂M; we choose the orientation of ∂M to be consistent with the orientation of M. Let ω be a $(d-1)$-form. Then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

To make sense of this very short statement, we need to put together everything on the previous six-and-a-half pages. We must remember that in $\mathbb{R}^3$, the exterior derivative $\mathrm{d}\omega$ will be gradient, or curl, or divergence. Then we must properly integrate the differential forms on both sides: for 1-forms, we'll get a flow or circulation integral over a curve, and for 2-forms, we'll get a flux integral across a surface.

If we do this, then what comes out of the theorem will be the appropriate version of one of the other integral theorems we've learned this semester:

- When M is a solid region in $\mathbb{R}^3$ and ω is a 2-form, we get the divergence theorem.
- When M is a surface in $\mathbb{R}^3$ and ω is a 1-form, we get Stokes' theorem.
- When M is a solid region in $\mathbb{R}^2$ and ω is a 1-form, we get Green's theorem.[3]

We can even generalize to curves, surfaces, and so forth in higher dimensions!

[2]For our purposes, a manifold is a curve, surface, or solid region in $\mathbb{R}^3$.

[3]Green's theorem has two forms: a tangent form and a normal form. But that's just a matter of interpreting the integrals: the actual equality between integrals is the same either way.

6 An example

Let's work through using the generalized Stokes' theorem on a specific example.

- Let M be the portion of the cone given by $z^2 = x^2 + y^2$ with $0 \leq z \leq 1$. We can parameterize this cone by $\mathbf{r}(u, v) = (u \cos v, u \sin v, u)$ where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$. (The orientation induced by this choice of $\mathbf{r}$ has a normal vector pointing upward and inward.)
- With this choice of M, the boundary ∂M is the circle $x^2 + y^2 = 1$ in the plane $z = 1$. We can parameterize this circle by $\mathbf{r}(t) = (\cos t, \sin t, 1)$ where $0 \leq t \leq 2\pi$ (with counterclockwise orientation, which is consistent with our orientation of M).
- Let's look at the vector field $\mathbf{F} = z^3\,\mathbf{i} + (x - z^3)\,\mathbf{j}$, which corresponds to the 1-form $\omega = z^3\,\mathrm{d}x + (x - z^3)\,\mathrm{d}y$.

First of all, let's integrate ω over the boundary ∂M. This should correspond to a counterclockwise circulation integral of $\mathbf{F}$.

To integrate $z^3\,\mathrm{d}x + (x - z^3)\,\mathrm{d}y$, we switch to t-coordinates from xyz-coordinates. Three things change:

- Rather than integrating over the circle ∂M in xyz-space, we integrate over the region $[0, 2\pi]$ in t-space. That's just an ordinary single-variable integral with respect to t.
- Using $\mathbf{r}(t) = (\cos t, \sin t, 1)$, we replace x by $\cos t$, y by $\sin t$, and z by 1.
- Using $\frac{\partial \mathbf{r}}{\partial t} = (-\sin t, \cos t, 1)$, we replace $\mathrm{d}x$ by $-\sin t\,\mathrm{d}t$, $\mathrm{d}y$ by $\cos t\,\mathrm{d}t$, and $\mathrm{d}z$ by 0.

This gives us

$$\int_{\partial M} z^3\,\mathrm{d}x + (x - z^3)\,\mathrm{d}y = \int_0^{2\pi} 1^3(-\sin t\,\mathrm{d}t) + (\cos t - 1^3)(\cos t\,\mathrm{d}t)$$
$$= \int_0^{2\pi} (\cos^2 t - \sin t - \cos t)\,\mathrm{d}t.$$

This is in fact the circulation integral of $\mathbf{F} = z^3\,\mathbf{i} + (x - z^3)\,\mathbf{j}$ around ∂M. If we write down $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\partial \mathbf{r}}{\partial t}\,\mathrm{d}t$, we get exactly the same integrand.

Now let's look at the other half of the theorem: the integral of $\mathrm{d}\omega$ over M. We know what the non-generalized Stokes' theorem says, so we expect a flux integral of the curl of $\mathbf{F}$ to come out of this.

First, we compute $\mathrm{d}\omega$; this is done earlier in the notes, where we got $\mathrm{d}\omega = 3z^2\,\mathrm{d}y\,\mathrm{d}z + 3z^2\,\mathrm{d}z\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}y$. So how do we find

$$\int_M 3z^2\,\mathrm{d}y\,\mathrm{d}z + 3z^2\,\mathrm{d}z\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}y?$$

Once again, we want to integrate in uv-coordinates, but everything in this integral is in xyz-coordinates. So we do a coordinate change, which has the following effects:

- An integral over M in xyz-space becomes an integral over $[0, 1] \times [0, 2\pi]$ in uv-space.
- According to the parameterization, we have $x = u \cos v$, $y = u \sin v$, and $z = u$.

- Taking the derivative, we get $\mathrm{d}x = \cos v\,\mathrm{d}u - u\sin v\,\mathrm{d}v$, $\mathrm{d}y = \sin v\,\mathrm{d}u + u\cos v\,\mathrm{d}v$, and $\mathrm{d}z = \mathrm{d}u$.
- The new thing compared to the line integral is that we want to take exterior products of these to understand terms like $\mathrm{d}y\,\mathrm{d}z$. We get

$$\begin{aligned}
\mathrm{d}y\,\mathrm{d}z &= (\sin v\,\mathrm{d}u + u\cos v\,\mathrm{d}v)(\mathrm{d}u) \\
&= -u\cos v\,\mathrm{d}u\,\mathrm{d}v, \\
\mathrm{d}z\,\mathrm{d}x &= (\mathrm{d}u)(\cos v\,\mathrm{d}u - u\sin v\,\mathrm{d}v) \\
&= -u\sin v\,\mathrm{d}u\,\mathrm{d}v, \\
\mathrm{d}x\,\mathrm{d}y &= (\cos v\,\mathrm{d}u - u\sin v\,\mathrm{d}v)(\sin v\,\mathrm{d}u + u\cos v\,\mathrm{d}v) \\
&= u\cos^2 v\,\mathrm{d}u\,\mathrm{d}v - u\sin^2 v\,\mathrm{d}v\,\mathrm{d}u \\
&= u\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Note that the expressions for $\mathrm{d}y\,\mathrm{d}z$, $\mathrm{d}z\,\mathrm{d}x$, and $\mathrm{d}x\,\mathrm{d}y$ are exactly the components of the cross product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$, multiplied by $\mathrm{d}u\,\mathrm{d}v$. When we include the components of $\mathrm{d}\omega$, we really do get the flux integral of the curl of $\mathbf{F}$.

After making all of these substitutions, we get

$$\begin{aligned}
\int_M \mathrm{d}\omega &= \int_M 3z^2\,\mathrm{d}y\,\mathrm{d}z + 3z^2\,\mathrm{d}z\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi}\int_0^1 3u^2(-u\cos v\,\mathrm{d}u\,\mathrm{d}v) + (3u^2)(-u\sin v\,\mathrm{d}u\,\mathrm{d}v) + (u\,\mathrm{d}u\,\mathrm{d}v) \\
&= \int_0^{2\pi}\int_0^1 (u - 3u^3\cos v - 3u^3\sin v)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

At this point, we have ended up at an ordinary iterated integral. (If we evaluate both integrals, we get π for both of them.)