Discrete Math Homework 6

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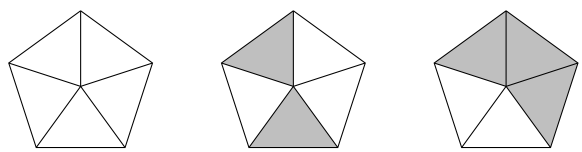
due Friday, March 31, 2023

1 Short answer

1. Divide a pentagon into 5 equal slices. Let A be the set of all $2^5 = 32$ ways to draw the polygon with some subset of the slices shaded (including all or none of them). A few elements of A are shown below:



Let R be the relation on A defined as follows: $x \mathrel{R} y$ if x can be rotated to turn it into y. This is an equivalence relation.

   (a) If x_1, x_2, and x_3 are the three elements of A drawn above, list the elements of the equivalence classes $[x_1]$, $[x_2]$, and $[x_3]$.

   (b) Find the total number of equivalence classes A has. (No combinatorial formula is necessary; you should be able to do this by counting.)

2. Evaluate the following expressions:

   (a) $\displaystyle\sum_{j=-1}^{1} \frac{j^2 - j + 1}{j^2 + j + 1}$.

   (b) $\displaystyle\sum_{k=1}^{10} a_k$ where $a_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$.

   (c) $\displaystyle\prod_{i=1}^{5} \frac{i}{i+3}$.

   (d) $\displaystyle\sum_{n=4}^{9} \sqrt{n}$ (simplify as much as possible).

3. Define a sequence $x_1, x_2, x_3, \ldots$ by the rule that $x_1 = 1$ and, for all $n \geq 1$, $x_{n+1} = 2n - x_n + 1$.

   (a) Compute the first 6 terms of the sequence.

(b) Guess a formula for x_n in terms of n.

2 Proofs

4. *You have already written a rough draft of this problem; now, read my feedback and write a final draft.*

   Define a function f from the positive integers $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \dots\}$ to the integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ by the following rule:

   - If $n \in \mathbb{Z}^+$ is odd, let $f(n) = \frac{n-1}{2}$.
   - If $n \in \mathbb{Z}^+$ is even, let $f(n) = -\frac{n}{2}$.

   Prove that f is injective (one-to-one).

5. *For this problem, write a rough draft of a proof; any reasonable attempt will be given full credit. I will give you feedback, and you will write a final draft on the next homework assignment.*

   Let $a_1, a_2, a_3, \dots$ satisfy the property that $a_n = \frac{a_{n-1}+a_{n+1}}{2}$ for all integers $n \geq 2$.

   Prove that if $a_1 = a_2$, then the sequence is constant (all terms of the sequence are equal).