Probability Theory Homework 4

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due Friday, March 3, 2023

1. Consider the following random experiment: there is a bag with 4 marbles in it, two red and two blue. We draw two marbles from the bag, without replacement.

   To make the sampling uniform, let's label the two red marbles R_1 and R_2 and the two blue marbles B_1 and B_2. Then the sample space is

   $$\Big\{R_1R_2,\ R_1B_1,\ R_1B_2,\ R_2B_1,\ R_2B_2,\ B_1B_2\Big\}.$$

   Finally, let $\mathbf{X}$ be the indicator variable equal to 1 if the two marbles drawn have the same color, and 0 if they have different colors.

   (a) Write down $\mathbf{X}$ as a function from the sample space to the real numbers.

   (b) What is $R_{\mathbf{X}}$, the range of $\mathbf{X}$?

   (c) Write down the probability mass function $P_{\mathbf{X}}$ of $\mathbf{X}$: this is a function from the range of $\mathbf{X}$ to the interval $[0, 1]$.

   (d) Identify the distribution of $\mathbf{X}$ by name, and give the parameter(s) of that distribution.

   (The functions in parts (a) and (c) of this problem both have only finitely many possible inputs. I don't care what notation you use in your answer, as long as you give me the value of the function for every possible input.)

2. A random variable $\mathbf{X}$ has range $R_{\mathbf{X}} = \{1, 2, 3, 4, 5, 6\}$ and probability mass function $P_{\mathbf{X}} \colon R_{\mathbf{X}} \to [0, 1]$ given by

   $$P_{\mathbf{X}}(k) = \begin{cases} c & 1 \le k \le 2 \\ 2c & 3 \le k \le 6. \end{cases}$$

   (a) Find the value of c for which this is a valid probability mass function.

   (b) Find $\Pr[\mathbf{X} \ge 4]$.

   (c) Find the expected value $\mathbb{E}[\mathbf{X}]$.

3. Alice and Bob play 5 rounds of a game. In each round, they both roll fair 6-sided dice. The player withe the highest result is the winner of that round (there could also be a tie).

   Each of the following quantities is a binomial distribution; give its parameters and expected value.

(a) The number of sixes rolled (by both players).

(b) The number of times that Alice rolls an odd number.

(c) The number of times that which Alice and Bob tie.

(d) The number of rounds in which Bob wins.

4. Suppose you want to flip a fair coin, but all you have is a biased coin that comes up heads with probability p. There is a sneaky trick for getting the fair coinflip you want anyway!

   The trick is: flip the coin twice, keeping track of the order. If the coin lands **first heads then tails**, output "heads" for the fair coinflip you want to simulate. If the coin lands **first tails then heads**, output "tails" for the fair coinflip you want to simulate. Otherwise, if the coin lands **on the same side twice**, ignore this result and start over from the beginning.

   (a) On a single iteration of this experiment, compute the probability of each result: output "heads", output "tails", or start over.

   (b) In terms of p, what is the distribution of the number of iterations required until you get an output? (Give the name and parameter(s) of the distribution.)

   (c) If $p = 0.01$ (the biased coin almost never lands heads), what is the expected number of times you'll have to flip the biased coin? (Keep in mind that we flip the coin twice on each iteration.)