Probability Theory Homework 6

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due Friday, March 31, 2023

1. Find the variance of a fair six-sided die whose sides are labeled $2, 3, 3, 3, 5, 5$.
2. Let $\mathbf{N} \sim \text{Binomial}(5, \frac{1}{2})$.
   (a) Find the conditional distribution $(\mathbf{N} \mid \mathbf{N} \text{ is even})$.
   (b) Find $\mathbb{E}[\mathbf{N} \mid \mathbf{N} \text{ is even}]$.
3. Suppose that $\mathbf{X}$ is uniformly chosen from $\{1, 2, 3\}$ and $(\mathbf{Y} \mid \mathbf{X} = k) \sim \text{Binomial}(2, \frac{k}{4})$.

   We have a 4-sided die, and paint $\mathbf{X}$ of its faces red, leaving the rest white; $\mathbf{X}$ is uniformly chosen from $\{1, 2, 3\}$. Then we roll the die twice, and let $\mathbf{Y}$ be the number of times a red face comes up. In other words, $(\mathbf{Y} \mid \mathbf{X} = k) \sim \text{Binomial}(2, \frac{k}{4})$.

   Give the following in the form of a table:

   (a) $P_{\mathbf{Y}|\mathbf{X}}(b \mid a)$, the conditional PMF of $\mathbf{Y}$ given $\mathbf{X}$.
   (b) $P_{\mathbf{XY}}(a, b)$, the joint PMF of $\mathbf{X}$ and $\mathbf{Y}$.
   (c) $P_{\mathbf{X}|\mathbf{Y}}(a \mid b)$, the conditional PMF of $\mathbf{X}$ given $\mathbf{Y}$.
   (d) Finally, explain what the $\mathbf{X} = 1$, $\mathbf{Y} = 1$ entry of your table in (c) means, in reference to the die with painted faces.
4. Two fair six-sided dice are rolled; let's say one is red and one is blue, just so that we can distinguish them. Find the expected value of the red die, given that the sum of the two dice is less than or equal to 5.
5. A group of n people stand in a line. On the count of three, each of them simultaneously chooses to look either left or right (with equal probability): at one of their neighbors.

   Let $\mathbf{X}$ be the number of *pairs* of adjacent people that end up facing each other. (For example, if $n = 5$ and the random facings are "Left, Left, Right, Left, Right" then only the 3^{rd} and 4^{th} people face each other.)

   (a) Find the expected value $\mathbb{E}[\mathbf{X}]$.
   (b) Find $\Pr[\mathbf{X} = 0]$.
   (c) Assuming n is even, find the maximum possible value of $\mathbf{X}$, and the probability that $\mathbf{X}$ is equal to that value.