Probability Theory Homework 7

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1. A fair die is rolled whose sides are labeled $2, 3, 3, 3, 5, 5$. Let $\mathbf{D}$ be the number that comes up.
   (a) Find the z-transform $\widehat{\mathbf{D}}(z)$ of $\mathbf{D}$, defined to be $\mathbb{E}[z^{\mathbf{D}}]$.
   (b) Let $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ be three independent copies of $\mathbf{D}$, and let $\mathbf{S} = \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3$. Find the z-transform $\widehat{\mathbf{S}}(z)$ by using your answer to (a).
   (c) Your answer to (b) should expand out to the polynomial
   $$\frac{z^{15}}{27} + \frac{z^{13}}{6} + \frac{z^{12}}{18} + \frac{z^{11}}{4} + \frac{z^{10}}{6} + \frac{11z^9}{72} + \frac{z^8}{8} + \frac{z^7}{24} + \frac{z^6}{216}.$$
   (I asked Mathematica.) Using this information, what is the probability $\Pr[\mathbf{S} \geq 12]$?

2. A random variable $\mathbf{X}$ has a probability density function of the form
   $$f_{\mathbf{X}}(t) = \begin{cases} Ct & 0 \leq t \leq 1 \\ C & 1 \leq t \leq 2 \\ C(3-t) & 2 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$
   (We can get such a random variable, for example, when we add an Uniform$(0,1)$ random variable to an independent Uniform$(0,2)$ random variable.)
   (a) Determine the value of C for which this is a probability density function.
   (b) Find the cumulative distribution function of $\mathbf{X}$.
   (c) Find $\Pr[0.5 \leq \mathbf{X} \leq 1.5]$.

3. Draw the cumulative distribution functions of the following distributions:
   (a) A continuous random variable chosen uniformly from the interval $[1, 6]$.
   (b) A continuous random variable chosen uniformly from the union $[1, 2] \cup [3, 4] \cup [5, 6]$.
   (c) A discrete random variable chosen uniformly from the set $\{1, 2, 3, 4, 5, 6\}$.

4. The random variables $\mathbf{X} \sim$ Exponential(1), $\mathbf{Y} \sim$ Uniform$(0, 2)$, and $\mathbf{Z}$ with the PDF
   $$f_{\mathbf{Z}}(x) = \begin{cases} \frac{2}{3} - \frac{2}{9}x & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$
   all have expected value 1. (We will learn how to find these expected values soon.)

   For each random variable, find the probability that it is less than its expected value of 1.