Probability Theory Homework 8

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due Friday, April 28, 2023

1. A random variable $\mathbf{Y}$ has the probability density function

$$f_{\mathbf{Y}}(t) = \begin{cases} 3t^{-4} & t \geq 1 \\ 0 & t < 1 \end{cases}$$

   (a) Find $\mathbb{E}[\mathbf{Y}]$.

   (b) Find $\mathrm{Var}[\mathbf{Y}]$.

2. If $\mathbf{U} \sim \mathrm{Uniform}(1, 3)$, find the probability density function of $\frac{1}{\mathbf{U}}$.

3. Let $\mathbf{Z} \sim \mathrm{Normal}(0, 1)$, and let $Q(1) = \Pr[\mathbf{Z} > 1]$. *(There is no closed form for $Q(1)$, but it is approximately 0.159.)*

   Find the conditional expectation $\mathbb{E}[\mathbf{Z} \mid \mathbf{Z} > 1]$ in terms of $Q(1)$.

4. You point your Geiger counter at a banana, waiting for it to click. You know that the time you have to wait is exponentially distributed with rate $\lambda = \frac{1}{10}$.

   Unfortunately, 10% of all Geiger counters have been sabotaged by Big Banana. A sabotaged Geiger counter, when pointed at a banana, will instead click after every 60 seconds to lull you into complacency.

   (a) Determine the CDF of the waiting time until your Geiger counter clicks (you do not know if your Geiger counter has been sabotaged).

   (b) If you have been waiting for 30 seconds and your Geiger counter still hasn't clicked yet, what is the probability that it has been sabotaged?

5. Let $\mathbf{U}$ and $\mathbf{V}$ be independent random variables, with the probability density functions

$$f_{\mathbf{U}}(s) = \begin{cases} \frac{1}{s^2} & s \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_{\mathbf{V}}(t) = \begin{cases} \frac{3}{8}t^2 & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   Find $\Pr[\mathbf{U} \leq \mathbf{V}]$.

6. *(The following problem is not for credit; this is the homework problem I'd assign for the material from the lecture on Monday, May 1^{st}. You don't have to write this up, but it's there for practice if you want it.)*

   A broken gumball vending machine spontaneously spits out gumballs of different colors. Over a short time scale, we can model this as a Poisson process which independently produces 0.1 red gumballs per second, 0.2 blue gumballs per second, and 0.3 yellow gumballs per second.

   Identify the distribution of the following quantities:

   (a) The number of blue gumballs produced in 1 minute.

   (b) The time until 5 yellow gumballs are produced.

   (c) The time interval between two consecutive gumballs produced (of any color).

   (d) The number of red gumballs produced after 10 total gumballs of all colors have been produced.