Probability Theory Homework 5

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1. You take a 20-question multiple choice exam on which every correct answer is worth 1 point, and every incorrect answer is worth $-\frac{1}{4}$ points to discourage guessing. Each question has five options: (A) through (E). On each question, you are able to eliminate one of the options as certainly wrong (leaving four), and then guess randomly between the other four options.

   (a) Let $\mathbf{X}$ be the number of correct answers you give. What distribution does $\mathbf{X}$ have (one of the named distributions we covered), and what are its parameters?

   (b) Express the number of points you receive as a linear transformation $a\mathbf{X} + b$ of $\mathbf{X}$, the random variable from part (a).

   (c) Find the probability that you get exactly 12.5 points.

2. A fair 12-sided die is rolled; let $\mathbf{D}$ be the number that comes up. Let $\mathbf{D} \bmod 5$ denote the remainder when $\mathbf{D}$ is divided by 5. (For example, if $\mathbf{D} = 5$, then $\mathbf{D} \bmod 5 = 0$, and if $\mathbf{D} = 8$, then $\mathbf{D} \bmod 5 = 3$.) Find the expected value $\mathbb{E}[\mathbf{D} \bmod 5]$.

3. Find the variance of a fair six-sided die whose sides are labeled $1, 2, 2, 3, 3, 3$.

4. For the sake of consistency, let's keep the same six-sided die with sides labeled $1, 2, 2, 3, 3, 3$, but the questions we ask about it here will be unrelated to the previous problem.

   We roll this six-sided die *three times*. Let $\mathbf{X}$ be the number of times the die lands 1; let $\mathbf{Y}$ be the number of times the die lands 2; let $\mathbf{Z}$ be the number of times the die lands 3.

   (a) Although there's three variables, it's enough to study the joint distribution of $\mathbf{X}$ and $\mathbf{Y}$, because $\mathbf{Z}$ is a function of $\mathbf{X}$ and $\mathbf{Y}$. What function? That is, what is $\mathbf{Z}$ in terms of $\mathbf{X}$ and $\mathbf{Y}$?

   (b) In the form of a 4×4 table, write down $P_{\mathbf{XY}}(a, b)$, the joint PMF of $\mathbf{X}$ and $\mathbf{Y}$.

   (c) In the form of a 4×4 table, write down $P_{\mathbf{X}|\mathbf{Y}}(a \mid b)$, the joint PMF of $\mathbf{X}$ given $\mathbf{Y}$.

   (d) Explain what the $\mathbf{X} = 1$, $\mathbf{Y} = 1$ entry of your table in (c) means in terms of our die-rolling experiment and the faces that come up.

5. (a) Find the conditional PMF of $(\mathbf{W} \mid 2 \le \mathbf{W} \le 6)$, where $\mathbf{W} \sim \text{Geometric}(p = \frac{1}{2})$.

   (b) Find the expected value $\mathbb{E}[\mathbf{W} \mid 2 \le \mathbf{W} \le 6]$.

6. At the Skittles factory, a bag of Skittles is filled by a mechanical scoop. The scoop picks up 9, 10, 11, or 12 Skittles (with equal probability of each number) and pours them into the bag; this is repeated a total of 5 times, resulting in a bag which contains between 45 and 60 Skittles.

   (a) Find $\text{Var}[\mathbf{S}]$, where $\mathbf{S}$ is the number of Skittles scooped up by the scoop.

   (b) Find $\text{Var}[\mathbf{B}]$, where $\mathbf{B}$ is the total number of Skittles in the bag.