Probability Theory Homework 6

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1. Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables with the distributions $\mathbf{X} \sim \text{Geometric}(p = \frac{1}{3})$ and $\mathbf{Y} \sim \text{Geometric}(p = \frac{2}{3})$. Find $\Pr[\mathbf{X} = \mathbf{Y}]$.

2. A fair die is rolled whose six sides are labeled $1, 2, 2, 3, 3, 3$. Let $\mathbf{D}$ be the number that comes up.

   (a) Find the z-transform $\widehat{\mathbf{D}}(z)$, defined to be $\mathbb{E}[z^{\mathbf{D}}]$.

   (b) Let $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4$ be four independent copies of $\mathbf{D}$ (that is, the results of four rolls of the die), and let $\mathbf{S} = \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3 + \mathbf{D}_4$. Find the z-transform $\widehat{\mathbf{S}}(z)$. (Don't expand.)

   (c) Wolfram Alpha told me that when you expand the answer to (b), you should get

   $$\frac{z^4}{1296} + \frac{z^5}{162} + \frac{z^6}{36} + \frac{13z^7}{162} + \frac{107z^8}{648} + \frac{13z^9}{54} + \frac{z^{10}}{4} + \frac{z^{11}}{6} + \frac{z^{12}}{16}.$$

   Using this information, what is $\Pr[\mathbf{S} \geq 10]$?

3. Let $\mathbf{T} \sim \text{Pascal}(m = 30, p = \frac{1}{2})$. For example, $\mathbf{T}$ could be measuring the number of coin tosses required to see 30 heads.

   Use an inequality covered in class to put an upper bound on the probability $\Pr[\mathbf{T} \geq 100]$. If you want to, you can try several approaches; for each additional (and sufficiently different) method you use, I will give a point of extra credit.

4. For each of the random variables below, plot the CDF.

   (a) A random real number chosen uniformly from the interval $[-2, 2]$.

   (b) A discrete random variable equal to -1 with probability $\frac{2}{3}$ and to 1 with probability $\frac{1}{3}$.

   (c) A random real number chosen uniformly from the set $[-2, -1] \cup [1, 2]$: the union of two intervals with a gap between them.