Discrete Math Homework 3

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due Friday, February 14, 2025

1 Logic and Quantifiers

1. Explain why the statements $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$ are not logically equivalent.

   (Hint: though a brute-force way to do this is with an 8-row truth table, that is not necessary.)

2. Let $\mathcal{F}$ be the following set (whose elements are themselves sets of integers):

$$\mathcal{F} = \Big\{\{1,2,3,4\}, \{2,4,5\}, \{1,3,4,6\}, \{4,5\}\Big\}$$

   For each of the following quantified statements, determine whether it is true or false.

   (a) $\forall S \in \mathcal{F}, \exists x \in \{1,2,3\}, x \in S$.

   (b) $\exists x \in \mathbb{Z}, \forall S \in \mathcal{F}, x \in S$.

   (c) $\exists S \in \mathcal{F}, \forall x \in \mathbb{Z}, x \in S$.

   (d) $\forall x \in \{1,2,3,4,5,6\}, \exists S \in \mathcal{F}, x \in S$.

2 Words and symbols

In this section, U will denote the universal set: the set of students at some university. S will denote the set of students on the soccer team, T will denote the set of students on the tennis team, and $F(x,y)$ will denote the predicate "x is friends with y".

3. Write the following statements using quantifier notation and logical operations.

   (a) Two people on the soccer team are always friends.

   (b) Nobody on the soccer team is friends with anyone on the tennis team.

   (c) Everyone on the tennis team has a friend at the university.

   (d) Some people have friends on both the soccer team and the tennis team.

   Note on commas: in symbolic form, these only belong after a quantified variable, such as when writing "$\forall x \in U, \dots$". Do not use them between two substatements: if you're tempted to write "p, q" then you probably mean either "$p \wedge q$" or "$p \to q$".

4. Write the following statements in plain English. It is not sufficient to literally translate every symbol; make the result a sentence a human would say.

   (a) $\exists x \in U, \forall y \in S, F(x, y)$.

   (b) $\forall x \in U, \exists y \in T, F(x, y)$.

   (c) $\exists x \in U, (\forall y \in S, F(x, y)) \vee (\forall z \in T, F(x, z))$.

   (d) $\forall x \in U, (\exists y \in T, F(x, y)) \rightarrow (\forall y \in T, F(x, y))$.

3 Proofs

1. *For this problem, write a rough draft of a proof; any reasonable attempt will be given full credit. I will give you feedback, and you will write a final draft on the next homework assignment.*

   Prove that for any two odd positive integers r and s, $3r - 5s$ is even.