Probability Theory Homework 1

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1. In a gambling game, you take a turn by rolling two four-sided dice with sides numbered 1, 2, 3, 4. You then put them together into a two-digit number by putting the smaller digit first; for example, if you roll a 3 and a 2, you get the number "23".

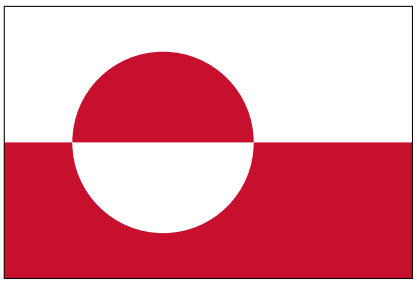
   (a) List the elements of the sample space in this random experiment: the possible two-digit numbers you can get.

   (b) Are we sampling uniformly from this sample space? Why or why not?

   (c) What is the probability that the two-digit number is 24 or greater?

2. There are 25 prime numbers between 1 and 100. If you choose a random number from 1 to 100, what is the probability that it is either prime or even? (There is only one even prime number, and that is 2.)

3. The flag of Greenland is shown below:



   It is a 12×18 rectangle; if placed on a coordinate plane with bottom left corner at $(0, 0)$ and top right corner at $(18, 12)$, it is divided in half by a line at $y = 6$ and has a circle of radius 4 centered at $(7, 6)$.

   The top half of the flag is white and the bottom half is red; within the circle, the two colors are swapped.

   Suppose that a point on this flag is chosen uniformly at random.

   (a) Find the probability that the chosen point is red.

   (b) Find the probability that the chosen point is red **or** inside the circle. (As usual, "or" in mathematics includes the possibility that both things happen.)

4. A random experiment has sample space $S = \{1, 2, 3, 4, \dots\}$ with $\Pr[\{k\}] = \frac{1}{k(k+1)}$ for all k.

   (a) Find $\Pr[\{1, 2, 3, 4\}]$.

   (b) Find $\Pr[\{3, 4, 5, 6, 7, \dots\}]$.

   (c) It turns out that $\Pr[\{1, 3, 5, 7, \dots\}] = \ln 2$. Use this fact to find $\Pr[\{6, 8, 10, 12, \dots\}]$.

5. Suppose that you know that your friend was born sometime in the year 2003, but not which day. Given this state of ignorance, we can model your friend's birthday as being chosen uniformly at random from the 365 days of a non-leap year. (In reality, these are not entirely uniform, but they're pretty close, and we can ignore this effect.)

   (a) What is the probability that your friend was born in January?

   (b) Suppose you remember that your friend's birthday is on the 20^{th} (of some month). Conditioned on this fact, what is the probability that your friend was born in January?

   (c) Now suppose you remember that your friend's birthday is on the 30^{th} (of some month). Conditioned on this fact, what is the probability that your friend was born in January?