# Probability Theory Homework 4

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due Friday, February 28, 2025

1. Each of the following random variables has a Binomial, Geometric, Pascal, or Hypergeometric distribution. Identify the distribution, and give its parameters.

   (a) You draw a hand of 5 cards from a standard 52-card deck. $\mathbf{A}$ is the number of aces you draw. (There are 4 aces in the deck.)

   (b) You are looking for a four-leaf clover in a clover field. The four-leaf mutation is quite rare, with an incidence rate of 0.1%. $\mathbf{C}$ is the number of clovers you will have to inspect before you find a four-leaf clover.

   (c) You receive many emails every day, but 90% of them are junk emails. (Let's assume that you don't have a spam filter to catch these junk emails.) Today, you receive 15 emails; $\mathbf{J}$ is the number of them that are junk.

   (d) Each booster pack of the famous card game *Sorcery: the Collecting* contains one mega-rare card, chosen uniformly from the 10 mega-rare cards in the set. Your goal is to get a full playset of 4 copies of the mega-rare card Purple Cabbage.

   You buy and open booster packs one at a time; $\mathbf{P}$ is the number of booster packs you will have to open in order to get four copies of Purple Cabbage.

2. A random variable $\mathbf{X}$ has range $R_{\mathbf{X}} = \{1, 2, 3, 4, 5, 6\}$ and probability mass function $P_{\mathbf{X}} \colon R_{\mathbf{X}} \to [0, 1]$ given by

$$P_{\mathbf{X}}(k) = \begin{cases} c & k \in \{1, 2\}, \\ 2c & k \in \{3, 4, 5, 6\}. \end{cases}$$

   (Watch out for a common mistake: this piecewise definition says that $P_{\mathbf{X}}(1) = c$ and that $P_{\mathbf{X}}(2) = c$, not that $\Pr[\mathbf{X} \in \{1, 2\}] = c$.)

   (a) Find the value of $c$ for which this is a valid probability mass function.

   (b) Find $\Pr[\mathbf{X} \geq 4]$.

   (c) Find the expected value $\mathbb{E}[\mathbf{X}]$.

3. Alice and Bob play 5 rounds of a game. In each round, they both roll fair 6-sided dice. The player withe the highest result is the winner of that round (there could also be a tie).

   Each of the following quantities is a binomial distribution; determine its parameters.

   (a) The number of sixes rolled (by both players).

(b) The number of times that Alice rolls an odd number.

(c) The number of times that which Alice and Bob tie.

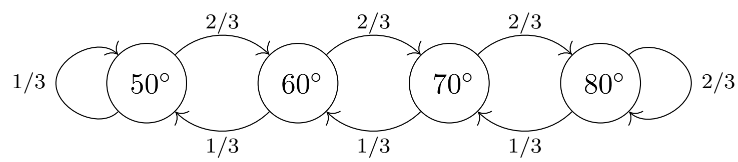
(d) The number of rounds in which Bob wins.

4. A basketball player is practicing free throws, and has a $\frac{1}{2}$ chance of making the first shot. However, the basketball player is very unconfident, so every shot after that strongly depends on the result of the previous throw:

   - After a successful throw, the basketball player has a $\frac{1}{2}$ chance of making the next shot and a $\frac{1}{2}$ chance of missing.
   - After a miss, the basketball player has only a $\frac{1}{3}$ chance of making the next shot, and a $\frac{2}{3}$ chance of another miss.

   Give a table of the probability that the basketball player makes the $k^{\text{th}}$ shot for $k = 1, 2, 3$.

5. The temperature (in Fahrenheit) in a certain region is always one of $\{50°, 60°, 70°, 80°\}$ and changes from day to day according to the following Markov chain:




   (That is, every day, there is a $\frac{2}{3}$ chance that it gets warmer—except that when it's 80°, the temperature just stays at 80°. Every day, there is a $\frac{1}{3}$ chance that it gets colder—except that when it's 50°, the temperature just stays at 50°.)

   Solve for the limiting probabilities $\pi_{50}, \pi_{60}, \pi_{70}, \pi_{80}$ of having each temperature, in the long run.

   *(Hint: try solving for $\pi_{60}, \pi_{70}, \pi_{80}$ in terms of $\pi_{50}$ first. Then determine what $\pi_{50}$ must be.)*