Probability Theory Homework 5

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1. 100 people are standing in a line, when suddenly it starts raining. Each person, independently, brought an umbrella with probability $\frac{1}{2}$. However, the umbrellas are very large: each umbrella covers the person holding it, and their neighbors on either side.

   Find the expected number of people that are not covered by anyone's umbrella.

2. At a university with 1000 students, 400 prefer reading and 600 prefer playing video games. The college newspaper conducts a poll of 50 students to ask them which activity they prefer.

   Define the following four random variables:

   - $\mathbf{W}$, the number of students polled who prefer reading.
   - $\mathbf{X}$, the number of students polled who prefer video games.
   - $\mathbf{Y}$, the number of students *not* polled who prefer reading.
   - $\mathbf{Z}$, the number of students *not* polled who prefer video games.

   (a) Give the distribution that $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ follow, and its parameters in each case.

   (b) Solve for $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ in terms of $\mathbf{W}$.

3. A "doubling cube", used in backgammon, is a fair 6-sided die whose sides are labeled with the numbers $2, 4, 8, 16, 32, 64$. Suppose that the doubling cube is rolled once, and $\mathbf{C}$ is the number that comes up.

   (a) Find the expected value $\mathbb{E}[\mathbf{C}]$.

   (b) Let $\mathbf{D}$ be the last digit of $\mathbf{C}$ (for example, if $\mathbf{C} = 16$, then $\mathbf{D} = 6$). Find $\mathbb{E}[\mathbf{D}]$.

4. Suppose that someone offers you a sequence of bets on the outcomes of rolling a die. Here is how the betting works.

   A prize pool starts out at \$1000. The die is rolled, and if it comes up 6, the prize pool is doubled, and the entire process is repeated. The game only ends once the number that comes up is not 6, at which point you win the amount in the prize pool.

   (For example, if the rolls are $6, 6, 6, 5$, the prize pool ends up doubling three times, and you win \$8000.)

   (a) Let $\mathbf{N} \sim \text{Geometric}(p = \frac{5}{6})$ be the number of times the die is rolled, and let $\mathbf{M}$ be the amount of money that you win. Write $\mathbf{M}$ as a function of $\mathbf{N}$.

(b) Find $\mathbb{E}[\mathbf{M}]$, the expected amount of money you win.

5. Suppose that the number of times that you go to the doctor each year has a Poisson distribution with an average of 2. Each visit to the doctor costs \$500.

   Your insurance will cover *all* of your medical fees if you end up going to the doctor 5 or more times. If you end up going to the doctor 4 or fewer times, you have to pay your medical fees yourself.

   (a) Find the expected amount of money you have to pay for medical fees in a year.

   (b) Find the conditional expected amount of money you have to pay for medical fees, given that you did not go to the doctor enough times to qualify for the insurance payout.