Probability Theory Homework 6

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1. At the Skittles factory, a bag of Skittles is filled by a mechanical scoop. The scoop picks up 9, 10, 11, or 12 Skittles (with equal probability of each number) and pours them into the bag; this is repeated a total of 5 times, resulting in a bag which contains between 45 and 60 Skittles.

   (a) Find $\text{Var}[\mathbf{S}]$, where $\mathbf{S}$ is the number of Skittles scooped up by the scoop.

   (b) Find $\text{Var}[\mathbf{B}]$, where $\mathbf{B}$ is the total number of Skittles in the bag.

2. Renting ice skates at a skating rink costs \$5.00. Rather than do this, Alice decides to buy cheap skates for \$25.00. However, the cheap skates have a $\frac{1}{10}$ chance of breaking down irreparably after every time Alice uses them.

   Once they break, Alice computes $\mathbf{Y}$, the amount of money she saved compared to renting skates an equal number of times. (Note that $\mathbf{Y}$ might be negative: if the skates break the first time, then $\mathbf{Y} = -20$, because it would have been 20 dollars cheaper to rent.)

   (a) What is the distribution of $\mathbf{X}$: the number of times Alice uses the cheap skates?

   (b) What is $\mathbf{Y}$, as a function of $\mathbf{X}$?

   (c) Find $\mathbb{E}[\mathbf{Y}]$ and $\text{Var}[\mathbf{Y}]$.

3. A children's toy has three lightbulbs, each controlled by a switch. A five-year-old child is playing with the toy. Each second, the child presses one of the three switches uniformly at random, which toggles one of the lightbulbs on or off.

   (a) Draw the transition diagram for a 4-state Markov chain representing this process, where the states are $\{0, 1, 2, 3\}$ and state k represents that k of the lightbulbs are turned on.

   (b) If the three lightbulbs all start out turned off, find the expected time, in seconds, until all three lightbulbs are turned on.

   (It is also possible to model this problem by an 8*-state Markov chain that keeps track of which lightbulbs are on. However, this would make the hitting time in part (b) harder to find.)*

4. We roll three fair dice; let $\mathbf{X}$ be the number of 1's rolled, and let $\mathbf{Y}$ be the number of 6's rolled. Find the covariance $\text{Cov}[\mathbf{X}, \mathbf{Y}]$.

5. Consider the following experiment. We choose a uniformly random word from the sentence

   "The quick brown fox jumped over the lazy dog."

Let $\mathbf{V}$ be the number of vowels and $\mathbf{C}$ be the number of consonants in the word chosen. (Here, we consider a, e, o, u, i, y to be vowels, and all other letters to be consonants.)

(a) Give the joint probability mass function of $\mathbf{C}$ and $\mathbf{V}$, in a table.

(b) Find the marginal distributions of $\mathbf{C}$ and $\mathbf{V}$.

(c) Suppose that instead $\mathbf{C}$ and $\mathbf{V}$ were computed for a uniformly random word from the English dictionary. Do you expect the covariance $\mathrm{Cov}[\mathbf{C}, \mathbf{V}]$ to be positive or negative? Explain why.

(I'm looking for an explanation that supports the answer you give, but it's okay if you've missed some aspect of the problem and your answer is wrong; I just want to know that you're thinking along the right lines, and that you are capable of explaining your mathematical reasoning in words.)