Probability Theory Homework 7

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1. Let $\mathbf{X} \sim \text{Geometric}(p = \frac{1}{3})$, $\mathbf{Y} \sim \text{Geometric}(p = \frac{1}{3})$, and $\mathbf{Z} \sim \text{Geometric}(p = \frac{2}{3})$, all independent. Which is more likely: that $\mathbf{X} = \mathbf{Y}$, or that $\mathbf{X} = \mathbf{Z}$? Why?

2. For each of the random variables below, plot the CDF.

   (a) A random real number chosen uniformly from the interval $[-2, 2]$.

   (b) A discrete random variable equal to -1 with probability $\frac{2}{3}$ and to 1 with probability $\frac{1}{3}$.

   (c) A random real number chosen uniformly from the set $[-2, -1] \cup [1, 2]$: the union of two intervals with a gap between them.

3. A random variable $\mathbf{A}$ has the probability density function

$$f_{\mathbf{A}}(t) = \begin{cases} c(2-t)\,\mathrm{d}t & 0 \le t \le 2 \\ c(t-3)\,\mathrm{d}t & 3 \le t \le 4 \\ 0\,\mathrm{d}t & \text{otherwise} \end{cases}$$

   for some constant c.

   (a) Find the value of c that makes $f_{\mathbf{A}}(t)$ a valid PDF.

   (b) Find the cumulative distribution function of $\mathbf{A}$.

   (c) Find $\Pr[1 \le \mathbf{A} \le 3]$.

4. The Pareto family of continuous distributions is not one of the ones that we will explore in detail, but it is worth seeing a little of. These have a probability density function of the form

$$f(t) = \begin{cases} at^b\,\mathrm{d}t & t \ge c \\ 0\,\mathrm{d}t & t < c \end{cases}$$

   except that not all values of the constants a, b, c are suitable.

   (a) Determine the two conditions that make this a valid PDF (there is an inequality that has to be satisfied, and a relationship between a, b, and c).

   (b) Find the expected value of this distribution as a function of a, b, and c (it will sometimes be $+\infty$).