Trigonometric Identities

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An **identity** is an equation containing one or more variables that is true for *all* values of the variables for which both sides of the equation are defined. The set of variables that is being used is either specified in the statement of the identity or is understood from the context. In this course, unless otherwise specified, we will assume that all variables under consideration are real numbers. Although our goal is to study identities that involve trigonometric functions, we will begin by giving a few examples of non–trigonometric identities so that we can become comfortable with the concept of what an identity is.

Example 1 *The equation*

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$

is an identity because the equation is true no matter what real numbers we substitute for a and b. For example, suppose that $a = 5$ and $b = -12$. We can easily check that it is true that

$$(5 + (-12))^2 = 5^2 + 2(5)(-12) + (-12)^2 .$$

(Both sides are equal to 49.)

We can prove that equation (1) is an identity by using elementary algebra (mainly the distributive property). The proof goes as follows: If a and b are any two real numbers, then

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= (a+b)a + (a+b)b \\ &= a^2 + ba + ab + b^2 \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2.\end{aligned}$$

Example 2 *The equation*

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

is an identity. To prove that it is, first note that

$$(a+b)^3 = (a+b)(a+b)^2$$

and then use the fact (which we have already proved in Example 1) that

$$(a+b)^2 = a^2 + 2ab + b^2.$$

By using this fact and again using the distributive property we obtain

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)\left(a^2+2ab+b^2\right) \\ &= a\left(a^2+2ab+b^2\right) + b\left(a^2+2ab+b^2\right) \\ &= a^3+2a^2b+ab^2+ba^2+2ab^2+b^3 \\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 \\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

Example 3 *The equation*

$$\frac{a}{a} = 1 \qquad (2)$$

is an identity because it is true for all real numbers, a, for which both sides of the equation are defined. For example, suppose that $a = -158$. We can easily see that the equation

$$\frac{-158}{-158} = 1$$

*is true. Note that the left hand side of equation (2) is not defined for **all** real numbers. In fact there is exactly one real number for which the equation is not defined and that is $a = 0$. (Recall that $0/0$ is not defined.) Nonetheless, equation (2) is still considered to be an identity because it is true for all real values of a for which both sides of the equation are defined.*

Example 4 *The equation*

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

is an identity. This equation is true for all values of θ *for which both sides of the equation are defined. Of course, there are many values of* θ *(such as* $\theta = \pi/2$ *and* $\theta = 5\pi/2$*) for which neither side of the equation is defined (due to the fact that division by* 0 *is not defined).*

Example 5 *Is the equation*

$$2x^2 - x = 2x^4 - x^3 \tag{3}$$

an identity? If it is, then it must be true no matter what real number we substitute for x*. Let us experiment a little:*

If we set $x = 0$*, then we obtain the statement*

$$2\,(0)^2 - 0 = 2\,(0)^4 - (0)^3$$

and this is certainly true.

Now let's try $x = 1$*. This gives us*

$$2\,(1)^2 - 1 = 2\,(1)^4 - (1)^3$$

which is also true (both sides are equal to 1*).*

Okay, now let's try $x = -1$*. This gives*

$$2\,(-1)^2 - (-1) = 2\,(-1)^4 - (-1)^3$$

which is also true because both sides are equal to 3*.*

It is starting to look like equation (3) might be identity, is it not? Let us try $x = 1/2$*. This gives*

$$2\left(\frac{1}{2}\right)^2 - \frac{1}{2} = 2\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^3$$

which is also true!

Does $x = 2$ *satisfy equation (3)? Let's see. Oops!*

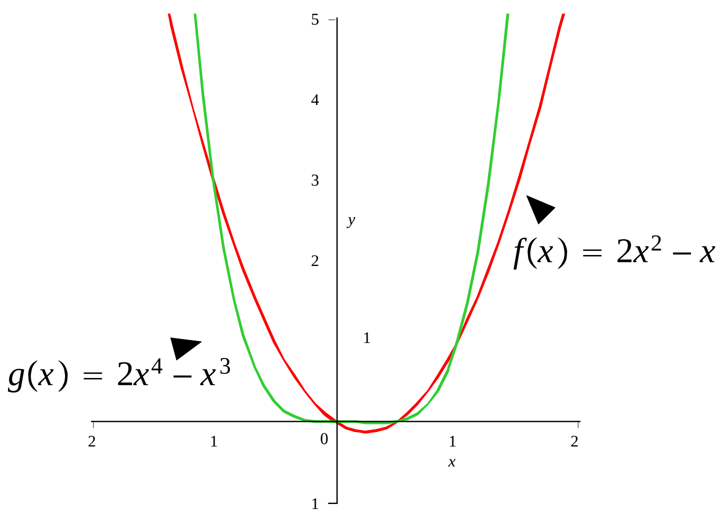
$$2\,(2)^2 - 2 = 2\,(2)^4 - (2)^3$$

is not true because the left hand side is equal to 6 *and the right hand side is equal to* 24*. This shows that equation (3) is not an identity.*

If we are presented with an equation of the form

$$f(x) = g(x)$$

that is **not** an identity, then it is usually pretty easy to prove that it is not an identity. All we need to do is to find some **single value** of x for which the equation $f(x) = g(x)$ is not true. In Example 5, the given equation was $f(x) = g(x)$ with $f(x) = 2x^2 - x$ and $g(x) = 2x^4 - x^3$. We discovered that $f(2) \neq g(2)$ which immediately allowed us to conclude that the equation $f(x) = g(x)$ is not an identity. Another way to arrive at this conclusion is to graph both of the functions $y = f(x)$ and $y = g(x)$. These graphs are shown below.



In order for $f(x) = g(x)$ to be an identity, the graphs of f and g must be exactly the same. Since the graphs of f and g shown above are not the same, then $f(x) = g(x)$ is not an identity. (Notice, however, that the graphs of f and g do intersect at the values $x = 0$, 1, -1, and $1/2$, which is why, in Example 5, we found that $f(x) = g(x)$ for these particular values of x.)

Now let us consider an example of a trigonometric identity.

Example 6 *The equation*

$$\cos(x - \pi) = -\cos(x) \tag{4}$$

is an identity. Why?

One way to see this quickly is to choose some strange random number such as $x = 368.83$ *and plug it into the equation (using a calculator set in radians mode). Using this strange number and a calculator we obtain*

$$\cos(368.83 - \pi) = 0.3023306652$$

and

$$-\cos(368.83) = 0.3023306652$$

which are the same! Of course this does not prove that equation (4) is an identity because we have only tried one value of x*. However, since the value of* x *that we tried was a strange random choice, what are the chances that we happened to just luckily stumble upon a value of* x *for which equation (4) is true? Since this would be very unlikely, we strongly suspect that equation (4) is an identity.*

Perhaps the next logical thing to do is to graph both of the functions $f(x) = \cos(x - \pi)$ *and* $g(x) = -\cos(x)$*. By using elementary transformations of* $y = \cos(x)$*, we observe that their graphs are identical! (See Figure 1.) This means that* $f(x) = g(x)$ *for all real numbers* x*.*

Let us summarize some ideas on how do decide whether or not a given equation is an identity. Suppose that the given equation is

$$f(x) = g(x).$$

1. Pick some random number (like $x = -681.5682$) and plug it into both sides of the equation. If the equation turns out to be true for this value of x, then chances are good that the equation is true for all values of x and hence that $f(x) = g(x)$ is an identity. If the equation is not true for this particular value of x, then we have proved that the equation $f(x) = g(x)$ is not an identity!

2. Graph both $y = f(x)$ and $y = g(x)$. If the graphs appear to be exactly the same, then there is a good chance that $f(x) = g(x)$ is an identity.

Notice that in the guidelines given above we used the phrases "chances are good" and "there is a good chance". That's because neither numerical substitution nor graphing can actually *prove* that a given equation is an identity. Graphing provides evidence (very strong evidence) of the truth of an identity but does not take the place of writing a proof. The type

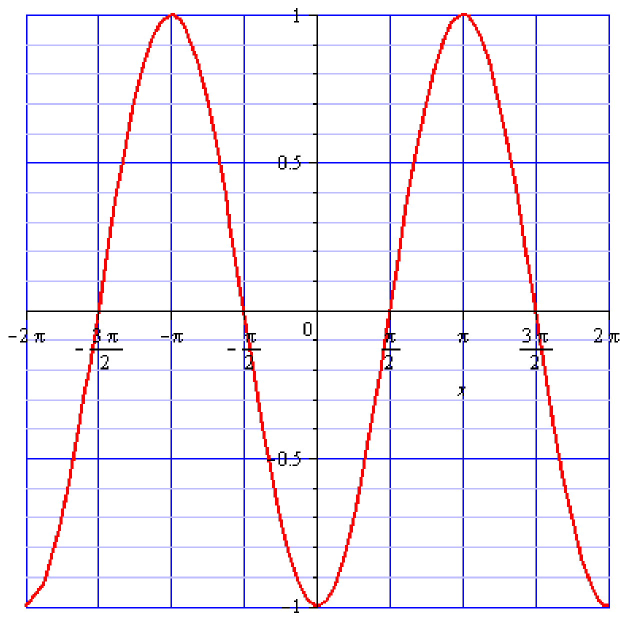


Figure 1: $y = \cos(x - \pi) = -\cos(x)$

of analytical reasoning that is needed to prove trigonometric identities is essential for the study of calculus and other higher topics in mathematics. In addition, the solutions of many types of applied problems require the use of trigonometric identities and the ability to manipulate these identities in order to obtain new identities and to solve trigonometric equations. These are the kinds of skills that one develops in studying trigonometric identities and their proofs in a trigonometry course such as this.

1 The Basic and Even–Odd Identities

Recall that if (x, y) is the point on the unit circle determined by a real number θ (see Figure 2), then the six trigonometric function values of θ are defined as follows:

$$\begin{array}{ll} \cos(\theta) = x & \sec(\theta) = 1/x \\ \sin(\theta) = y & \csc(\theta) = 1/y \\ \tan(\theta) = y/x & \cot(\theta) = x/y. \end{array}$$

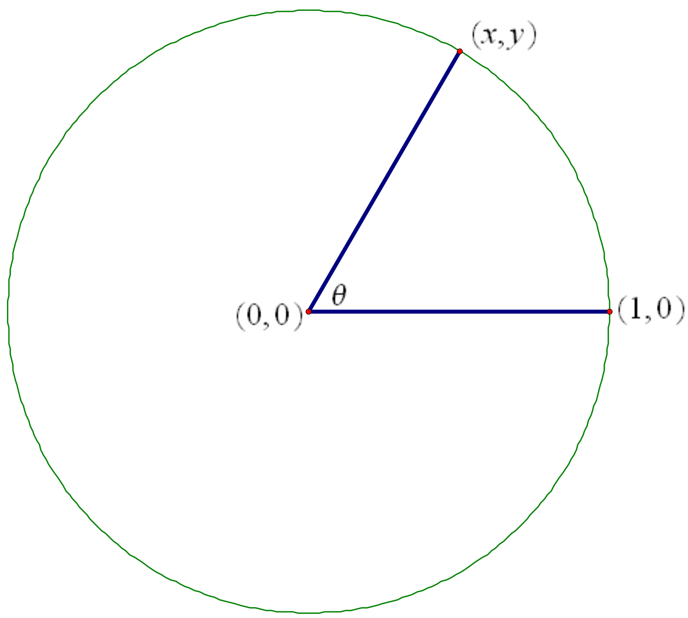


Figure 2: Unit circle

The definitions of the six trigonometric functions lead immediately to some identities which we will call the basic identities.

The Basic Identities

$\csc(\theta) = 1/\sin(\theta)$	$\sin(\theta) = 1/\csc(\theta)$
$\sec(\theta) = 1/\cos(\theta)$	$\cos(\theta) = 1/\sec(\theta)$
$\cot(\theta) = 1/\tan(\theta)$	$\tan(\theta) = 1/\cot(\theta)$
$\cot(\theta) = \cos(\theta)/\sin(\theta)$	$\tan(\theta) = \sin(\theta)/\cos(\theta)$

The basic identities are very easy to prove. For example, to prove the identity

$$\csc(\theta) = \frac{1}{\sin(\theta)},$$

we just note that if (x, y) is the point on the unit circle corresponding to the number θ and $y \neq 0$ then

$$\csc(\theta) = \frac{1}{y} \qquad \text{and} \qquad \sin(\theta) = y.$$

It follows immediately that

$$\csc(\theta) = \frac{1}{\sin(\theta)}.$$

(Note that this identity makes sense only for values θ for which $\sin(\theta) \neq 0$.)

Another basic set of identities are the even–odd identities:

The Even–Odd Identities

$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$
$$\cot(-\theta) = -\cot(\theta)$$
$$\sec(-\theta) = \sec(\theta)$$
$$\csc(-\theta) = -\csc(\theta)$$

The even–odd identities for the sine and cosine functions can be proved by looking at the unit circle (see Figure 3) and observing that for any θ we have

$$\cos(-\theta) = x = \cos(\theta)$$

and

$$\sin(-\theta) = -y = -\sin(\theta)$$

(Although Figure 3 only illustrates the case that the point (x, y) is in the first quadrant, we can easily check the other cases by drawing some additional pictures.)

Another way to convince ourselves of the truth of the even–odd identities for sine and cosine is to look at the graphs of these functions. The graph of $y = \cos(\theta)$ is symmetric with respect to the y axis (meaning that the cosine function is even and hence that $\cos(-\theta) = \cos(\theta)$ for all real numbers θ). The graph of $y = \sin(\theta)$ is symmetric with respect to the origin (meaning that the sine function is odd and hence that $\sin(-\theta) = -\sin(\theta)$ for all real numbers θ).

To prove the other even–odd identities, we can use the basic identities and the already established even–odd identities for sine and cosine. For example, here is how we prove the identity

$$\tan(-\theta) = -\tan(\theta):$$

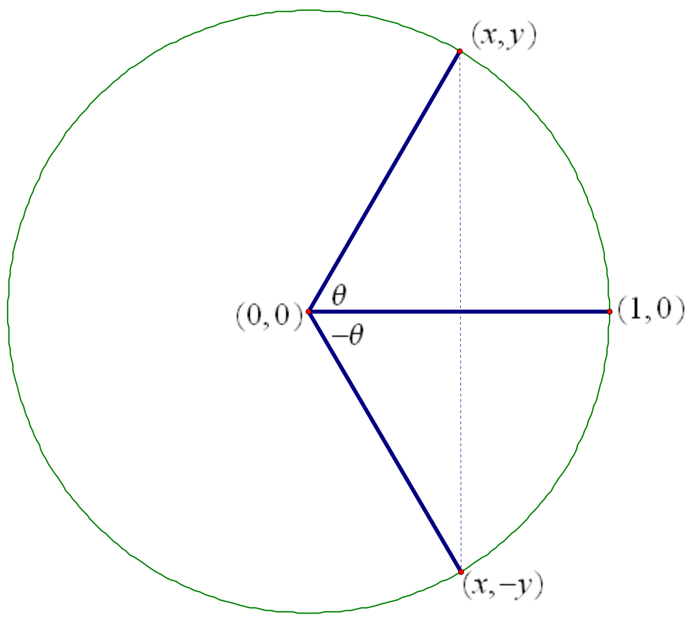


Figure 3: Symmetry of the Unit Circle

Proof.

$$\begin{aligned}
\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} && \text{(basic identity)} \\
&= \frac{-\sin(\theta)}{\cos(\theta)} && \text{(even–odd identities for sine and cosine)} \\
&= -\frac{\sin(\theta)}{\cos(\theta)} && \text{(basic algebra)} \\
&= -\tan(\theta) && \text{(basic identity)}
\end{aligned}$$

■

Observe that the even–odd identities tell us that the functions cosine and secant are even functions; whereas the functions sine, cosecant, tangent, and cotangent are odd functions.

2 The Pythagorean Identities

By looking at Figure 4, we observe that *every* point (x, y) on the unit circle satisfies the equation

$$x^2 + y^2 = 1.$$

This is due to the Pythagorean Theorem! (Although Figure 4 illustrates this fact only for points (x, y) in the first quadrant, we can see that it is true for (x, y) in the other quadrants by drawing a few additional pictures.)

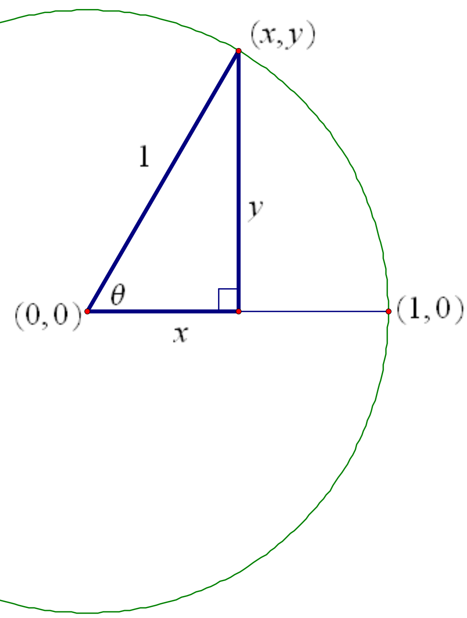


Figure 4: Fundamental Pythagorean Identity

Now suppose that (x, y) is the point on the unit circle that corresponds to a given real number θ. Since $\cos(\theta) = x$ and $\sin(\theta) = y$, we see that

$$(\cos(\theta))^2 + (\sin(\theta))^2 = x^2 + y^2 = 1.$$

The identity that we have just proved,

$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1,$$

is called the *Fundamental Pythagorean Identity.* (The reason for its name is obviously because it is a result of the Pythagorean Theorem.)

There are two other Pythagorean Identities that we will list, but before doing so we make a comment about some notation that is commonly used in trigonometry: When raising a trigonometric function value to a power, for example, $(\cos(\theta))^2$, we often write $\cos^2(\theta)$ instead of writing $(\cos(\theta))^2$. Either way of writing this is correct. Writing it as $(\cos(\theta))^2$ is actually "more correct" in that it captures the true meaning of what is going on. When we write x^2 we mean x times x. That is, $x^2 = x \cdot x$. Thus $(\cos(\theta))^2$ means $\cos(\theta) \cdot \cos(\theta)$. Nonetheless, since the alternate notation is so commonly used in mathematics, me must get used to it. $\cos^2(\theta)$ is also taken to mean $\cos(\theta) \cdot \cos(\theta)$. With this convention in mind, we can write the Fundamental Pythagorean Identity as

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

The three Pythagorean identities are listed below.

The Pythagorean Identities

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

The second two Pythagorean identities listed above are easily proved by using the first one. Here is the proof of the identity

$$1 + \tan^2(\theta) = \sec^2(\theta):$$

Proof. Beginning with the (already proved) identity

$$\cos^2(\theta) + \sin^2(\theta) = 1,$$

we divide both sides of this equation by $\cos^2(\theta)$ to obtain

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}.$$

This gives us

$$\frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

which is the same as

$$1+\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2=\left(\frac{1}{\cos(\theta)}\right)^2.$$

By using some basic identities, we observe that this is the same as

$$1+(\tan(\theta))^2=(\sec(\theta))^2.$$

This completes the proof. (Recall that $\tan^2(\theta)$ is just another way of writing $(\tan(\theta))^2$ and $\sec^2(\theta)$ is just another way of writing $(\sec(\theta))^2$.) ■

3 The Sum and Difference Identities

The sum and difference identities for the sine and cosine functions are identities involving two variables. These identities allow us to express the sine or cosine of a sum or difference of two numbers, α and β, in terms of sines and cosines of the individual numbers. These identities are given below.

Sum and Difference Identities for Sine and Cosine

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) \tag{5}$$

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \tag{6}$$

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) \tag{7}$$

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \tag{8}$$

In order to prove the sum and difference identities, we will need to use the *distance formula* which is used to compute the distance between two points in the plane. Referring to Figure 5, we see that if (x_1, y_1) and (x_2, y_2) are two points in the plane (and these two points do not lie on the same horizontal or vertical line), then we can form a right triangle whose $90°$ angle occurs at the point (x_2, y_1). By the Pythagorean Theorem, the distance, d, between the points (x_1, y_1) and (x_2, y_2) satisfies

$$d^2=(x_2-x_1)^2+(y_2-y_1)^2$$

and hence

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

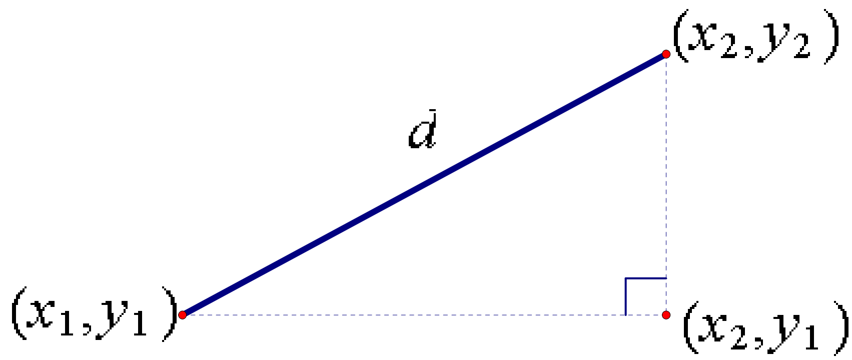


Figure 5: The Distance Formula

Either one of the above two formulas are referred to as the distance formula. In proving the sum and difference identities, we will find it convenient to work with the first form (so as to avoid square roots).

To begin, we will prove the difference identity

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

This proof relies on Figures 6 and 7 in which we assume that $\alpha > \beta > 0$. (Other cases can be considered by drawing additional pictures but we will not do that here.) The important thing to observe in comparing Figures 6 and 7 is that the length d is **the same** in each figure. This is because the chord with length d is determined by the angle $\alpha - \beta$ in each figure.

In Figure 6, we see, by the distance formula, that

$$\begin{aligned} d^2 &= (\cos(\beta) - \cos(\alpha))^2 + (\sin(\beta) - \sin(\alpha))^2 \\ &= \cos^2(\beta) - 2\cos(\alpha)\cos(\beta) + \cos^2(\alpha) \\ &+ \sin^2(\beta) - 2\sin(\alpha)\sin(\beta) + \sin^2(\alpha) \\ &= \cos^2(\beta) + \sin^2(\beta) + \cos^2(\alpha) + \sin^2(\alpha) \\ &- 2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta) \\ &= 2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)). \end{aligned}$$

On the other hand, in Figure 7 we see that

$$\begin{aligned}d^2 &= (\cos(\alpha-\beta)-1)^2 + (\sin(\alpha-\beta)-0)^2 \\ &= \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta) \\ &= 2 - 2\cos(\alpha-\beta)\,.\end{aligned}$$

It must therefore be true that

$$2 - 2\left(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\right) = 2 - 2\cos(\alpha-\beta)\,.$$

After performing some elementary algebra on the above equation we obtain

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

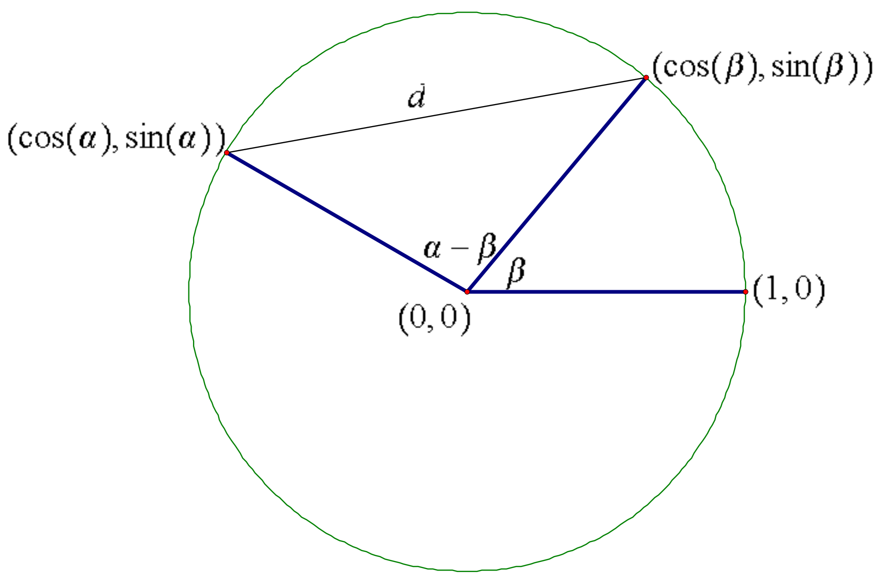


Figure 6: For proof of cosine difference identity

Having proved identity (5), we can now use the even–odd identities to prove identity (6). In particular, since identity (5) holds for **all** real numbers α and β, then identity (5) allows us to conclude that if α and β are any real

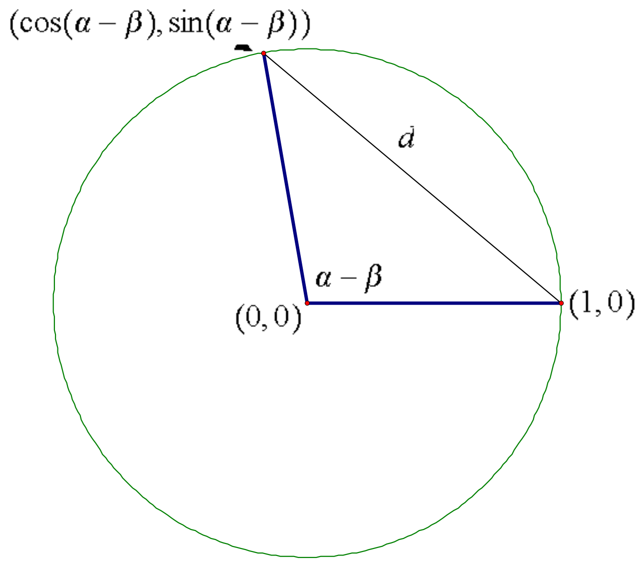


Figure 7: For proof of cosine difference identity

numbers, then

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha-(-\beta)) \\ &= \cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta)+\sin(\alpha)(-\sin(\beta)) \quad \text{(even–odd identities)} \\ &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).\end{aligned}$$

This proves identity (6).

In order to prove the sum and difference identities for sine, we will use the cofunction identities.

The Cofunction Identities

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta) \tag{9}$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta) \tag{10}$$

To prove the cofunction identity (9) we apply identity (5) as follows:

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta)\\ &= (0)\cos(\theta)+(1)\sin(\theta)\\ &= \sin(\theta)\,.\end{aligned}$$

To prove the cofunction identity (10) we observe that

$$\begin{aligned}\cos(\theta) &= \cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right)\\ &= \cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}-\theta\right)+\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}-\theta\right)\\ &= (0)\cos\left(\frac{\pi}{2}-\theta\right)+(1)\sin\left(\frac{\pi}{2}-\theta\right).\end{aligned}$$

We can now prove the sum and difference identities for sine. First we will prove identity (7):

$$\begin{aligned}\sin(\alpha-\beta) &= \cos\left(\frac{\pi}{2}-(\alpha-\beta)\right)\\ &= \cos\left(\left(\frac{\pi}{2}-\alpha\right)+\beta\right)\\ &= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)-\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta)\\ &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\,.\end{aligned}$$

Finally we prove identity (8):

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha-(-\beta))\\ &= \sin(\alpha)\cos(-\beta)-\cos(\alpha)\sin(-\beta)\\ &= \sin(\alpha)\cos(\beta)-\cos(\alpha)(-\sin(\beta))\\ &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\,.\end{aligned}$$

By using the sum and difference identities for sine and cosine, we can also derive sum and difference identities for tangent, cotangent, secant, and cosecant. Since the most frequently used of these remaining four are the ones for tangent, we will give them here and prove one of them.

Sum and Difference Identities for Tangent

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \tag{11}$$

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)} \tag{12}$$

Here is the proof of identity (11).

Proof.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \text{ (basic identities)}\\
&= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \text{ (sum identities)}\\
&= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}\left(\frac{1}{1-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}\right) \text{ (factoring)}\\
&= \left(\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}\right)\left(\frac{1}{1-\tan(\alpha)\tan(\beta)}\right)\\
&= (\tan(\alpha)+\tan(\beta))\left(\frac{1}{1-\tan(\alpha)\tan(\beta)}\right)\\
&= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.
\end{aligned}$$

■

4 The Double Angle Identities

The double angle identities are special cases of the sum and difference identities in which α and β are both equal to the same number θ. In this case, $2\theta = \theta+\theta = \alpha+\beta$ and we obtain identities for expanding $\sin(2\theta)$ and $\cos(2\theta)$.

The Double Angle Identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \tag{13}$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \tag{14}$$
$$\cos(2\theta) = 1 - 2\sin^2(\theta) \tag{15}$$
$$\cos(2\theta) = 2\cos^2(\theta) - 1 \tag{16}$$
$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} \tag{17}$$

We will give the proofs of the first three identities and then leave the proofs of the fourth and fifth as exercises.

Here is the proof of the identity (13).

Proof.

$$\begin{aligned}\sin(2\theta) &= \sin(\theta+\theta)\\ &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)\\ &= 2\sin(\theta)\cos(\theta)\end{aligned}$$

■

Here is the proof of the identity (14).

Proof.

$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)\\ &= \cos^2(\theta) - \sin^2(\theta)\end{aligned}$$

■

Here is the proof of the identity (15).

Proof.

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \ \text{(by the previously proved identity)}\\ &= \left(1-\sin^2(\theta)\right) - \sin^2(\theta) \ \text{(Pythagorean Identity)}\\ &= 1 - 2\sin^2(\theta).\end{aligned}$$

■

5 Product to Sum and Sum to Product Identities

Suppose we write down both the sum and difference identities for cosine:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

and then add these two equations. We thus obtain

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha)\cos(\beta). \tag{18}$$

This equation can also be written as

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right). \tag{19}$$

The identity (19) is called the *product to sum* identity for cosine.
Now, in equation (18), suppose that we make the substitutions

$$x = \alpha + \beta$$
$$y = \alpha - \beta.$$

This gives

$$x + y = 2\alpha$$
$$x - y = 2\beta$$

and thus

$$\alpha = \frac{x+y}{2}$$
$$\beta = \frac{x-y}{2}.$$

We can thus rewrite equation (18) in terms of x and y as follows:

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right). \tag{20}$$

Equation (20) is called the *sum to product* identity for cosine.

By similar reasoning, we can derive all of the following product to sum and sum to product identities.

Product to Sum Identities

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$
$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$$

Sum to Product Identities

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos(y) + \cos(x) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos(y) - \cos(x) = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

6 The Logic of Proving Identities

What are the correct ways to write the proof of an identity $f(x) = g(x)$?

Method 1 We can begin by writing down one side of the identity (either $f(x)$ or $g(x)$) and then writing a string of equalities that leads to the other side. Such a proof looks like this:

$$f(x) = f_1(x) = f_2(x) = \ldots = g(x).$$

Method 2 We can find an expression that $f(x)$ and $g(x)$ are both equal to. Such a proof looks like this:

$$f(x) = h(x) \tag{21}$$
$$g(x) = h(x) \tag{22}$$

and therefore

$$f(x) = g(x).$$

Most likely, the proof of identities (21) and (22) will require a proof using Method 1.

Here is an example of proof that uses Method 1.

Example 7 *Prove the identity*

$$\frac{1+\cos(2\theta)}{\sin(2\theta)} = \cot(\theta).$$

Proof.

$$\begin{aligned}\frac{1+\cos(2\theta)}{\sin(2\theta)} &= \frac{1+(2\cos^2(\theta)-1)}{2\sin(\theta)\cos(\theta)} \text{ (double angle identities)} \\ &= \frac{2\cos^2\theta}{2\sin(\theta)\cos(\theta)} \text{ (simplification)} \\ &= \frac{\cos(\theta)}{\sin(\theta)} \text{ (simplification)} \\ &= \cot(\theta) \text{ (basic identity)}\end{aligned}$$

■

Here is an example of a proof that uses Method 2 (and Method 1 in proving each part of the Method 2 proof).

Example 8 *Prove the identity*

$$\frac{\sin(x)-\cos(x)}{\cos^2(x)} = \frac{\tan^2(x)-1}{\sin(x)+\cos(x)}.$$

Proof. First we work with the left hand side of the identity.

$$\begin{aligned}
\frac{\sin(x)-\cos(x)}{\cos^2(x)} &= \frac{\sin(x)-\cos(x)}{\cos^2(x)} \cdot \frac{\sin(x)+\cos(x)}{\sin(x)+\cos(x)} \text{ (multiplication by 1)} \\
&= \frac{(\sin(x)-\cos(x))(\sin(x)+\cos(x))}{\cos^2(x)(\sin(x)+\cos(x))} \text{ (multiplication of fractions)} \\
&= \frac{\sin^2(x)-\cos^2(x)}{\cos^2(x)\sin(x)+\cos^3(x)} \text{ (simplification)}
\end{aligned}$$

Now we work with the right hand side of the identity.

$$\begin{aligned}
\frac{\tan^2(x)-1}{\sin(x)+\cos(x)} &= \frac{\frac{\sin^2(x)}{\cos^2(x)}-1}{\sin(x)+\cos(x)} \text{ (basic identity)} \\
&= \frac{\frac{\sin^2(x)}{\cos^2(x)}-1}{\sin(x)+\cos(x)} \cdot \frac{\cos^2(x)}{\cos^2(x)} \text{ (multiplication by 1)} \\
&= \frac{\sin^2(x)-\cos^2(x)}{\cos^2(x)\sin(x)+\cos^3(x)} \text{ (multiplication of fractions).}
\end{aligned}$$

We now see that each side of the identity that we want to prove is equal to the expression

$$\frac{\sin^2(x)-\cos^2(x)}{\cos^2(x)\sin(x)+\cos^3(x)}.$$

Therefore we have proved the identity

$$\frac{\sin(x)-\cos(x)}{\cos^2(x)} = \frac{\tan^2(x)-1}{\sin(x)+\cos(x)}.$$

■

There is an **incorrect** way to write a proof of an identity that is often used by students.

This is an incorrect proof of the identity $f(x) = g(x)$:

Incorrect Method of Proof

$$
\begin{aligned}
f(x) &= g(x) \\
&\implies f_1(x) = g_1(x) \\
&\implies f_2(x) = g_2(x) \\
&\implies \cdots \\
&\vdots \\
&\implies \cdots \\
h(x) &= h(x).
\end{aligned}
$$

The above method of proof is incorrect because it's very first line is $f(x) = g(x)$ which is the identity that we want to prove! If we begin by assuming that $f(x) = g(x)$ is already true, then what need is there to write any more? Just because the final line of this "proof" is $h(x) = h(x)$, which is a true statement, we cannot conclude that the statement $f(x) = g(x)$ is also true. True statements always lead to true statements but false statements can **also** lead to true statements! Here is an example to illustrate why the incorrect proof method is incorrect.

Example 9 *Prove that*

$$1 = -1.$$

Incorrect Proof.

$$
\begin{aligned}
1 &= -1 \\
&\implies (1)^2 = (-1)^2 \text{ (squaring both sides)} \\
&\implies 1 = 1 \text{ (simplification)}.
\end{aligned}
$$

Therefore $1 = -1$ is true!

Well, obviously we know that $1 = -1$ is not true. This illustrates why this method of proof cannot be used. ∎

Here is one more example in which we use the incorrect method of proof to "prove" a true identity.

Example 10 *Prove the identity*

$$\frac{1 - \sin(\theta)}{\cos(\theta)} = \frac{\cos(\theta)}{1 + \sin(\theta)}. \tag{23}$$

Incorrect Proof.

$$\frac{1-\sin(\theta)}{\cos(\theta)} = \frac{\cos(\theta)}{1+\sin(\theta)}$$
$$\implies (1-\sin(\theta))(1+\sin(\theta)) = \cos^2(\theta) \text{ (by cross multiplying)}$$
$$\implies 1-\sin^2(\theta) = \cos^2(\theta) \text{ (simplification – difference of two squares)}$$
$$\implies \sin^2(\theta)+\cos^2(\theta) = 1 \text{ (rearrangement).}$$

The final statement

$$\sin^2(\theta)+\cos^2(\theta) = 1$$

is a statement that we know to be true. It is the fundamental Pythagorean identity. This might lead us to incorrectly believe that we have proved the original identity (23). However the proof is not correct. The identity (23) **is** in fact true, but we have not supplied a correct proof of it. We will now give a correct proof of (23). ■

Proof.

$$\begin{aligned}\frac{1-\sin(\theta)}{\cos(\theta)} &= \frac{1-\sin(\theta)}{\cos(\theta)}\cdot\frac{1+\sin(\theta)}{1+\sin(\theta)}\\ &= \frac{1-\sin^2(\theta)}{\cos(\theta)(1+\sin(\theta))}\\ &= \frac{\cos^2(\theta)}{\cos(\theta)(1+\sin(\theta))} \text{ (fundamental Pythagorean identity)}\\ &= \frac{\cos(\theta)}{1+\sin(\theta)} \text{ (simplification)}\end{aligned}$$

■