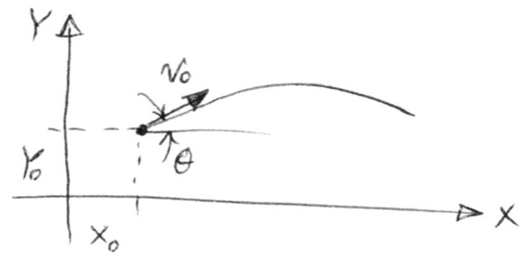
# Motion of a Projectile (12.6)

Dynamics - S. Nasseri

The motion can be treated as 2 rectilinear motions:

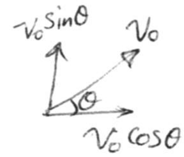
- ⊛ One in the horizontal direction experiencing zero acceleration.
- ⊛ One in the vertical direction experiencing constant acceleration. (gravity)

**Case 1** A projectile launched at point $(x_0, y_0)$:




Consider the components of the initial velocity $V_0$

$$\begin{cases} V_{ox} = V_0 \cdot \cos\theta & (1) \\ V_{oy} = V_0 \cdot \sin\theta & (2) \end{cases}$$




Horizontal Motion: Position in the x direction depends only on $V_{ox}$ and $t$, $a = 0$

So: $V = V_0 + at \rightarrow V_x = V_{ox}$ at any time the horizontal component is equal to $V_0 \cos\theta$.

$$x = x_0 + V_0 t + \frac{1}{2} a t^2 \longrightarrow \boxed{x = x_0 + V_{ox} t} \quad (3)$$

Vertical Motion: $a = \text{constant} = -g \Rightarrow$

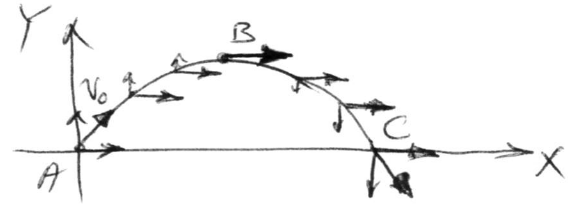
$$\begin{cases} V_y = V_{oy} - gt & (4) \\ y = y_0 + V_{oy} \cdot t - \frac{1}{2} g t^2 & (5) \\ V_y^2 = V_{oy}^2 - 2g(y - y_0) & (6) \end{cases}$$

(from 4 & 5)

plug (1) and (2) in these equations:

$$\begin{cases} x = x_0 + V_0 \cos\theta \cdot t & (7) \\ V_y = V_0 \sin\theta - gt & (8) \\ y = y_0 + V_0 \sin\theta t - \frac{1}{2} g t^2 & (9) \\ V_y^2 = (V_0 \sin\theta)^2 - 2g(y - y_0) & (10) \end{cases}$$

**Case 2** $\quad x_0 = 0 \;,\; y_0 = 0$




consider equations 7 to 10:

$V_{ox}$ = constant

from A to B $\rightarrow$ $V_{oy}$ decreases

from B to C $\rightarrow$ $V_{oy}$ increases

$$\begin{cases} x = V_0 \cos\theta\, t & (11) \\ V_y = V_0 \sin\theta - g t & (12) \\ y = V_0 \sin\theta\, t - \frac{1}{2} g t^2 & (13) \\ V_y^2 = (V_0 \sin\theta)^2 - 2 g y & (14) \end{cases}$$

find the "path equation":

This is infact expressing $y$ in terms of $x$ and $\theta$:

find $t$ from (11) and plug in (13):

$$y = V_0 \sin\theta \left(\frac{x}{V_0 \cos\theta}\right) - \frac{g}{2}\left(\frac{x}{V_0 \cos\theta}\right)^2 \Rightarrow \boxed{y = V_0 \cdot x \cdot \tan\theta - \frac{g x^2}{2 V_0^2}\left(1 + \tan^2\theta\right)} \quad (15)$$

Use $\boxed{\frac{1}{\cos^2\theta} = 1 + \tan^2\theta}$

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What is the maximum height ($h$) and the total distance traveled?

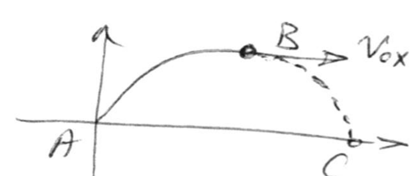
What is the total time for this projectile to get to point C?

- for total time, find the time which takes to reach point B and them multiply this time by 2:

At point B: $\begin{cases} V_x = V_{ox} \\ V_y = 0 \end{cases}$ just the horizontal component

from (12) $\rightarrow$ $V_y = V_0 \sin\theta - g t_B \longrightarrow 0 = V_0 \sin\theta - g t_B$

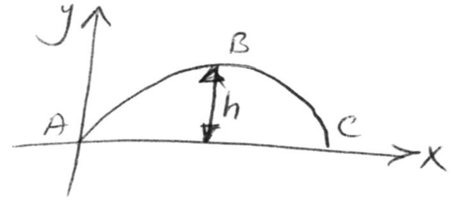
$$t_B = \frac{V_0 \sin\theta}{g}$$




$$\boxed{t_{total} = 2 t_B = \frac{2 V_0 \sin\theta}{g}} \quad (16)$$

$t_{total} = t_C$

Finding $h$:



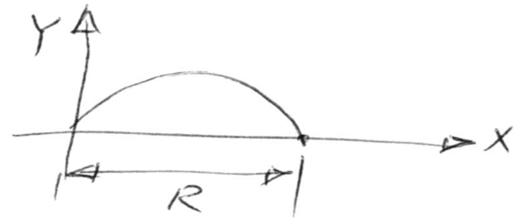

$h$ is in fact $y$ at $t = t_B$:

plug $t_A = \dfrac{V_0 \sin\theta}{g}$ in eq. (13):

$$y\Big|_{t=t_B} = h = V_0 \sin\theta \cdot t_B - \tfrac{1}{2} g t_B^2 = V_0 \sin\theta \left(\frac{V_0 \sin\theta}{g}\right) - \tfrac{1}{2} g \left(\frac{V_0^2 \sin^2\theta}{g^2}\right)$$

$$\longrightarrow h = \frac{(V_0 \sin\theta)^2}{g} - \frac{1}{2}\frac{(V_0 \sin\theta)^2}{g} \Rightarrow \boxed{h = \frac{V_0^2 \sin^2\theta}{2g}} \quad (17)$$

Finding $R$:




plug the total time in (11)

$$R = x\Big|_{t=t_C=t_{total}} = V_0 \cos\theta \cdot t_{total} = V_0 \cos\theta \cdot \left(\frac{2 V_0 \sin\theta}{g}\right)$$

$$\longrightarrow \boxed{R = \frac{2 V_0^2 \sin\theta \cos\theta}{g}} \quad (18)$$

What is the angle at which this distance would be maximized:

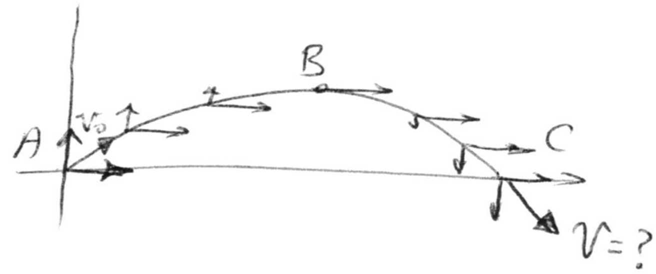
when $\theta = 45^\circ \longrightarrow R = \text{Maximum}$

$$\sin\theta = \cos\theta = \frac{\sqrt{2}}{2} \longrightarrow \boxed{R_{Max} = 0.031\, V_0^2} \quad (19)$$

what is the velocity when the projectile hits the ground?

$V_x = \text{constant} = V_0 \cos\theta$

$V_y = V_0 \sin\theta - gt$




at $t = t_{total} = \dfrac{2V_0 \sin\theta}{g} \Longrightarrow V_y = V_0 \sin\theta - g\left(\dfrac{2V_0 \sin\theta}{g}\right)$

$$\boxed{V_y = -V_0 \sin\theta}$$

$$V_c = V_{final} = \sqrt{(V_0 \cos\theta)^2 + (-V_0 \sin\theta)^2} = V_0$$

So the final velocity is equal to $V_0$ (initial velocity), just the direction has changed.


$$\boxed{V_{final} = V_0}$$ 20