Putnam practice - Number theory problems

1. True or false: If $p \geq 5$ is a prime number, then 24 divides $p^2 - 1$ without remainder.

2. Show that there are no integers a, b, and c such that $a^2 + b^2 - 8c = 6$.

3. (2003 A1) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers $n = a_1 + a_2 + \cdots + a_k$ with k positive and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$?

4. (2014 A1) Prove that every non-zero coefficient of the Taylor series of $(1 - x + x^2)e^x$ about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

5. (2005 A1) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$ where r and s are non-negative integers and no summand divides another.

6. (2024 A1) Determine all positive integers n for which there exist positive integers a, b, c such that $2a^n + 3b^n = 4c^n$.

7. (2013 A2) Let S be the set of non-perfect squares. For $n \in S$, consider choices of integers $n < a_1 < a_2 < \ldots < a_r$ such that $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. (e.g., $2 \cdot 3 \cdot 6 = 6^2$ and $f(2) = 6$.) Show that $f : S \to \mathbb{Z}$ is one-to-one.