

Putnam practice - Probability problems solutions

1. What is the probability that a 5-digit number starts or ends with a 1?

The total number of 5-digit number is $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 90000$.

Let A be the set of 5-digit number starting with 1. Then the size of A is $1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000$.

Let B be the set of 5-digit number ending with 1. Then the size of B is $9 \cdot 10 \cdot 10 \cdot 10 \cdot 1 = 9000$.

But A and B overlap. Their overlap $A \cap B$ has size $1 \cdot 10 \cdot 10 \cdot 10 \cdot 1 = 1000$.

So, the probability is $\frac{10000+9000-1000}{90000} = \frac{18000}{90000} = \frac{1}{5}$.

Another way to do this is by complement C which are 5-digit numbers neither starting or ending with 1.

Then the size of C is $8 \cdot 10 \cdot 10 \cdot 9 = 72000$.

Then the probability is $\frac{90000-72000}{90000} = \frac{18000}{90000} = \frac{1}{5}$ again.

2. A fair coin is flipped 5 times, what is the probability that there is no consecutive heads?

Let x_n be the number of different outcomes with no consecutive heads under n flips.

Case 1: An outcome can start with T . Then the number of different outcomes with no consecutive heads under $n - 1$ flips is x_{n-1} .

Case 2: An outcome can start with H . Then the second flip must be a T for we cannot have consecutive heads. After that, the number of different outcomes with no consecutive heads for the remaining $n - 2$ flips is x_{n-2} .

Hence, we derive a recursion $x_n = x_{n-1} + x_{n-2}$. Note that $x_1 = 2$ and $x_2 = 3$.

So, $x_3 = x_2 + x_1 = 3 + 2 = 5$, $x_4 = x_3 + x_2 = 5 + 3 = 8$, and $x_5 = x_4 + x_3 = 8 + 5 = 13$.

Therefore, the probability is $\frac{13}{2^5} = \frac{13}{32}$.

3. (2001 A2) You have coins $C_1, C_2, \dots, C_n$. For each k , C_k is biased such that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If n coins are tossed, what is the probability that the number of heads is odd?

You can start by inspecting small cases when $n = 1, 2, 3$.

With 1 coin, the probability of getting an odd number of head is the same as getting 1 head. So, the answer is $\frac{1}{2(1)+1} = \frac{1}{3}$.

With 2 coins, we want either HT or TH which has probability $\frac{1}{3} \cdot (1 - \frac{1}{5}) + (1 - \frac{1}{3}) \cdot \frac{1}{5} = \frac{2}{5}$.

I will let you try 3 coins on your own.

One arrives at the conjecture that the probability of having an odd number of heads with n coins is $\frac{n}{2n+1}$.

Then one can prove this by induction.

Base step: When $n = 1$, the probability of getting an odd number of head is the same as getting 1 head. So, the answer is $\frac{1}{2(1)+1} = \frac{1}{3}$ which matches with $\frac{n}{2n+1}$ when $n = 1$.

Induction step: Suppose it is true for some $n \geq 1$. We want to prove that it is true for $n + 1$.

Case 1: Suppose the $n + 1$ -th coin flip is a head. Then we need an even number of heads from the previous n coin flips. Hence, the probability is $\frac{1}{2(n+1)+1} \cdot (1 - \frac{n}{2n+1}) = \frac{n+1}{(2n+1)(2n+3)}$.

Case 2: Suppose the $n + 1$ -th coin flip is a tail. Then we need an odd number of heads from the previous n coin flips. Hence, the probability is $(1 - \frac{1}{2(n+1)+1}) \cdot \frac{n}{2n+1} = \frac{(2n+2)n}{(2n+1)(2n+3)}$.

Combining them, we have the probability is $\frac{n+1}{(2n+1)(2n+3)} + \frac{2(n+1)n}{(2n+1)(2n+3)} = \frac{(n+1)(1+2n)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3} = \frac{n+1}{2(n+1)+1}$ which completes the induction step.

4. (2004 A1) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N , but by the end of the season $S(N)$ was more than 80% of N . Was there necessarily a moment in between when $S(N)$ was exactly 80% of N ?

The answer is **yes**. We will prove it by contradiction. Suppose the contrary that there is no moment in between when $S(N)$ was exactly 80% of N .

Since $S(N)$ was less than 80% of N , but by the end of the season $S(N)$ was more than 80% of N , there must be a last moment M where $S(N)$ switched from below 80% to above 80% and stays above till the end of the season.

Hence, $S(M) < 0.8M$ and $S(M + 1) > 0.8(M + 1)$ (we suppose no equality occurs).

Clearly, $S(M + 1) = S(M)$ or $S(M) + 1$ as Shanille can miss or make the $M + 1$ -th free throw.

However, $S(M + 1) = S(M)$ cannot be the case as that would imply $0.8(M + 1) < S(M + 1) = S(M) < 0.8M$ or $0.8M + 0.8 < 0.8M$ which is false.

Therefore, we must have $S(M + 1) = S(M) + 1$.

Thus, $S(M) < \frac{4M}{5}$ and $S(M) + 1 > \frac{4(M+1)}{5}$.

Multiplying everything by 5, we get $5S(M) < 4M$ and $5S(M) + 5 > 4M + 4$.

Consequently, $4M - 1 = 4M + 4 - 5 < 5S(M) < 4M$ which is a contradiction as there is no integer between two consecutive integers $4M - 1$ and $4M$.

Therefore, there was necessarily some moment when $S(N)$ was exactly 80% of N .

5. (2014 A4) Suppose X is a random variable that takes on only non-negative integer values with expected values $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. Determine the smallest possible value of the probability of the event $X = 0$.

The smallest value $P(X = 0)$ is $1/3$.

For convenience, let $p_i = P(X = i)$ for $i = 0, 1, 2, \dots$

From definition of probability and expected value, we have

$$\sum_{i=0}^{\infty} p_i = 1, \quad \sum_{i=0}^{\infty} i \cdot p_i = 1, \quad \sum_{i=0}^{\infty} i^2 \cdot p_i = 2, \quad \sum_{i=0}^{\infty} i^3 \cdot p_i = 5. \quad (1)$$

We can organize the equations into matrix form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & ..1.. & 1 \\ 0 & 1 & 2 & 3 & 4 & ..i.. & 1 \\ 0 & 1 & 4 & 9 & 16 & ..i^2.. & 2 \\ 0 & 1 & 8 & 27 & 64 & ..i^3.. & 5 \end{pmatrix}$$

We perform row operations $R_3 - R_2$, $R_4 - R_2$, $R_4 - 3R_3$, $R_3/2$, $R_4/6$, and get

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & ..1.. & 1 \\ 0 & 1 & 2 & 3 & 4 & ..i.. & 1 \\ 0 & 0 & 2 & 6 & 12 & ..i^2 - i.. & 1 \\ 0 & 0 & 6 & 24 & 60 & ..i^3 - i.. & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & ..1.. & 1 \\ 0 & 1 & 2 & 3 & 4 & ..i.. & 1 \\ 0 & 0 & 2 & 6 & 12 & ..i^2 - i.. & 1 \\ 0 & 0 & 0 & 6 & 24 & ..i^3 - 3i^2 + 2i.. & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & ..1.. & 1 \\ 0 & 1 & 2 & 3 & 4 & ..i.. & 1 \\ 0 & 0 & 1 & 3 & 6 & ..\frac{i(i-1)}{2}.. & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 4 & ..\frac{i(i-1)(i-2)}{6}.. & \frac{1}{6} \end{pmatrix}$$

Hence, we have

$$p_0 = 1 - p_1 - p_2 - p_3 - p_4 - \dots - p_i - \dots \quad (2)$$

$$p_1 = 1 - 2p_2 - 3p_3 - 4p_4 - \dots - ip_i - \dots \quad (3)$$

$$p_2 = \frac{1}{2} - 3p_3 - 6p_4 - \dots - \frac{i(i-1)}{2}p_i - \dots \quad (4)$$

$$p_3 = \frac{1}{6} - 4p_4 - \dots - \frac{i(i-1)(i-2)}{6}p_i - \dots \quad (5)$$

Substituting the equation (3) into (2), we get

$$p_0 = p_2 + 2p_3 + 3p_4 + \dots + (i-1)p_i + \dots \quad (6)$$

Substituting the equation (4) into (6), we get

$$p_0 = \frac{1}{2} - p_3 - 2p_4 - \dots - \frac{(i-1)(i-2)}{2}p_i - \dots \quad (7)$$

Finally, substituting the equation (5) into (7), we get

$$p_0 = \frac{1}{3} + 2p_4 + \dots + \frac{(i-1)(i-2)(i-3)}{6}p_i + \dots \geq \frac{1}{3}.$$

Equality can be achieved by having $p_4 = p_5 = p_6 = \dots = 0$, $p_3 = \frac{1}{6}$, $p_2 = 0$, $p_1 = \frac{1}{2}$, and $p_0 = \frac{1}{3}$. I will let you check that the equations in (1) are satisfied.

Another way to solve this problem is by means of **probability generating function** (by Tony Qiao):

$$F(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + \dots = \sum_{i=0}^{\infty} p_i x^i \quad \text{with} \quad F(1) = \sum_{i=0}^{\infty} p_i = 1.$$

Note that $F(x)$ converges for $|x| \leq 1$ by comparison test with $\sum_{i=0}^{\infty} p_i = 1$. Then

$$F'(x) = \sum_{i=0}^{\infty} ip_i x^{i-1} \quad \text{with} \quad F'(1) = \sum_{i=0}^{\infty} ip_i = E[X] = 1.$$

Taking derivative again, we have

$$F''(x) = \sum_{i=0}^{\infty} i(i-1)p_i x^{i-2} \quad \text{with} \quad F''(1) = \sum_{i=0}^{\infty} i(i-1)p_i = E[X^2] - E[X] = 2 - 1 = 1.$$

Taking derivative one more time, we get

$$F'''(x) = \sum_{i=0}^{\infty} i(i-1)(i-2)p_i x^{i-3} \quad \text{with} \quad F'''(1) = \sum_{i=0}^{\infty} i(i-1)(i-2)p_i = E[X^3] - 3E[X^2] + 2E[X] = 1.$$

Now, by Taylor series of $F(x)$ with remainder term, for any $0 \leq x \leq 1$, there exists $c \in [x, 1]$ such that

$$F(x) = F(1) + \frac{F'(1)}{1!}(x-1) + \frac{F''(1)}{2!}(x-1)^2 + \frac{F'''(1)}{3!}(x-1)^3 + \frac{F^{(4)}(c)}{4!}(x-1)^4$$

Hence,

$$p_0 = F(0) = F(1) - \frac{F'(1)}{1!} + \frac{F''(1)}{2!} - \frac{F'''(1)}{3!} + \frac{F^{(4)}(c)}{4!}.$$

Since $F(x)$ has non-negative coefficients and $c \geq 0$, we must have $F^{(4)}(c) \geq 0$. Therefore,

$$p_0 \geq 1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$