

Geometry and Synthetic Geometry

Problem 1 (Sphere and Hemisphere). *Given any five points on the surface of a sphere, prove that some four of them lie on a single closed hemisphere.*

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Problem 3 (Rational Points on a Circle). *A point in the plane is called rational if both of its coordinates are rational numbers. What is the maximum number of rational points that can lie on a circle whose center is not a rational point?*

Problem 4 (Cube in a Cone). *A right circular cone has base radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube lies in the base of the cone. Find the side length of the cube.*

Problem 5 (Napoleon-Type Equilateral Construction). *Let A, B, C be the vertices of an equilateral triangle in the plane, and let P be any point. Construct points A', B', C' so that the triangles PAA' , PBB' , and PCC' are equilateral and lie externally to P . Show that A', B', C' also form an equilateral triangle, and that its center coincides with P .*

Problem 6 (Simson Line). *Let A, B, C be points on a circle with center O , and let P be any point on that circle. From P , drop perpendiculars to the sides BC , CA , and AB , meeting them at A' , B' , and C' , respectively. Prove that the points A' , B' , and C' are collinear.*