Putnam practice - Generating Functions

Given a sequence of numbers a_n, the corresponding generating function is the power series:

$$\sum_{n=0}^{\infty} a_n x^n.$$

Example 1: Suppose $a_n = 1$ for all n. By the geometric series formula, we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{1}$$

when $|x| < 1$.

Example 2: By differentiating (1), we obtain

$$\sum_{n=0}^{\infty} n x^{n-1} = \sum_{m=0}^{\infty} (m+1) x^m = \frac{1}{(1-x)^2}.$$

Hence, the generating function for $a_n = n$ is

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

Application: The infinite sum

$$1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} = \sum_{n=0}^{\infty} \frac{n+1}{2^n} = \frac{1}{(1-1/2)^2} = 4.$$

Putnam (2015): Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

Solution: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the corresponding generating function.

The recurrence relation gives $a_n - 4a_{n-1} + a_{n-2} = 0$ for $n \geq 2$.

Consider $x^2 A(x) - 4xA(x) + A(x)$. Its generating function is

$$\sum_{n=0}^{\infty} a_n x^{n+2} - 4 \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n.$$

Renaming the labels, it becomes

$$\sum_{m=2}^{\infty} a_{m-2} x^m - 4 \sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (a_{n-2} - 4a_{n-1} + a_n) x^n - 4a_0 x + a_0 + a_1 x = 1 - 2x.$$

Therefore,

$$(x^2 - 4x + 1) A(x) = 1 - 2x \quad \text{or} \quad A(x) = \frac{1-2x}{x^2 - 4x + 1}.$$

By quadratic formula, the quadratic $x^2-4x+1=(x-(2+\sqrt{3}))(x-(2-\sqrt{3}))$. We can break up

$$\frac{1-2x}{x^2-4x+1}=\frac{B}{x-(2+\sqrt{3})}+\frac{C}{x-(2-\sqrt{3})}$$

into partial fractions. In other words,

$$1-2x=B(x-(2-\sqrt{3}))+C(x-(2+\sqrt{3})).$$

Putting in $x=2+\sqrt{3}$ and $x=2-\sqrt{3}$ respectively and solving, we obtain

$$B=\frac{-3-2\sqrt{3}}{2\sqrt{3}}=-\frac{2+\sqrt{3}}{2}\quad\text{and}\quad C=\frac{-3+2\sqrt{3}}{-2\sqrt{3}}=-\frac{2-\sqrt{3}}{2}.$$

Hence,

$$\begin{aligned}A(x)=&-\frac{2+\sqrt{3}}{2}\cdot\frac{1}{x-(2+\sqrt{3})}-\frac{2-\sqrt{3}}{2}\cdot\frac{1}{x-(2-\sqrt{3})}\\=&\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2+\sqrt{3}}}+\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2-\sqrt{3}}}\\=&\frac{1}{2}\cdot\frac{1}{1-(2-\sqrt{3})x}+\frac{1}{2}\cdot\frac{1}{1-(2+\sqrt{3})x}\end{aligned}$$

by using conjugate. Applying the geometric seriest formula, we obtain

$$\sum_{n=0}^{\infty}a_nx^n=\frac{1}{2}\sum_{n=0}^{\infty}((2-\sqrt{3})x)^n+\frac{1}{2}\sum_{n=0}^{\infty}((2+\sqrt{3})x)^n$$

when $|(2+\sqrt{3})x|<1$. Comparing coefficients of x^n on both sides, we have the nice formula

$$a_n=\frac{1}{2}(\alpha^n+\beta^n)$$

where $\alpha=2+\sqrt{3}$ and $\beta=2-\sqrt{3}$.

Notice that, for odd positive integer k,

$$\frac{a_{kn}}{a_n}=\frac{\alpha^{kn}+\beta^{kn}}{\alpha^n+\beta^n}=\alpha^{(k-1)n}-\alpha^{(k-2)n}\beta^n+\cdots-\alpha^n\beta^{(k-2)n}+\beta^{(k-1)n}.$$

The left hand side is a rational number since a_{kn} and a_n are positive integers. The right hand side is of the form $A+B\sqrt{3}$ for some integers A and B after expanding powers of α and β out.

Since $\sqrt{3}$ is irrational and the two sides are the same, it follows that $B=0$ and the ratio $\frac{a_{kn}}{a_n}=A$ is an integer.

In particular, with $n=5$ and $k=403$, we obtain a_{2015} is divisible by a_5.

One can calculate recursively that $a_2=4a_1-a_0=4(2)-1=7$, $a_3=4a_2-a_1=4(7)-2=26$, $a_4=4a_3-a_2=4(26)-7=97$, $a_5=4a_4-a_3=4(97)-26=362=2\cdot 181$.

As $\sqrt{181}<\sqrt{196}=14$, one can check that 181 is an odd prime number as $181\equiv 1\pmod 2$, $181\equiv 1\pmod 3$, $181\equiv 1\pmod 5$, $181\equiv 6\pmod 7$, $181\equiv 5\pmod{11}$, $181\equiv 12\pmod{13}$.

Therefore, $181 \mid a_5$ and $a_5 \mid a_{2015}$ imply that the odd prime 181 is a factor of a_{2015}.

Exercise 1:

Use the generating function method to prove the following formula for the Fibonacci sequence:

$$F_n = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \frac{1}{\sqrt{5}}\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Exercise 2:

Putnam (1999) Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.

Hint: First, show that the quadratic $1 - 2x - x^2$ factors to be $(1 - (1 - \sqrt{2})x)(1 - (1 + \sqrt{2})x)$.

Then decompose $\frac{1}{1-2x-x^2}$ into partial fractions.

Apply geometric series formula to show that $a_n = \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}]$.

Use algebra to show that $m = 2n + 2$ works.