

Putnam practice - Introduction

(2024 A1) Determine all positive integers n for which there exist positive integers a, b, c such that $2a^n + 3b^n = 4c^n$.

One can test things out with small n 's and apply basic principles like even and odd integer consideration and rearranging the equation.

However, one needs to write up a clear proof / explanation to receive full credit.

Answer: $n = 1$ is the only solution.

Proof:

For $2a + 3b = 4c$, we have a solution $a = 1, b = 2, c = 2$ for example.

We will prove that $2a^n + 3b^n = 4c^n$ has no positive integer solution when $n \geq 2$.

We prove this by contradiction. Suppose the contrary that the equation has some positive integer solution.

Without loss of generality, we may assume that $\gcd(a, b, c) = 1$.

For if $d = \gcd(a, b, c) > 1$, say $a = da_1, b = db_1, c = dc_1$. Then we have $2(da_1)^n + 3(db_1)^n = 4(dc_1)^n$ which yields $2a_1^n + 3b_1^n = 4c_1^n$, a solution with $\gcd(a_1, b_1, c_1) = 1$. i.e., any integer solution can be reduced to one with the greatest common divisor among a, b, c being one.

Rearrange the equation, we have $3b^n = 4c^n - 2a^n = 2(2c^n - a^n)$.

Clearly, the right hand side is even. Hence, the left hand side is also even.

However, since 3 is odd, b must be even. Say $b = 2b_1$.

Then the original equation becomes $2a^n + 3 \cdot 2^n \cdot b_1^n = 4c^n$ or $a^n + 3 \cdot 2^{n-1} \cdot b_1^n = 2c^n$.

Then we have $a^n = 2c^n - 3 \cdot 2^{n-1} \cdot b_1^n = 2(c^n - 3 \cdot 2^{n-2} \cdot b_1^n)$.

The above right hand side is even. So, the left hand side is also even and a must be even. Say $a = 2a_1$.

Substituting back to the original equation, we get $2 \cdot 2^n a_1^n + 3 \cdot 2^n \cdot b_1^n = 4c^n$.

This implies $2^{n-1}a_1^n + 3 \cdot 2^{n-2} \cdot b_1^n = c^n$.

Case 1: If $n > 2$, then the above left hand side $= 2(2^{n-2}a_1^n + 3 \cdot 2^{n-3} \cdot b_1^n)$ which is even.

This implies the right hand side is also even and c must be even. Say $c = 2c_1$.

But then $\gcd(a, b, c) = \gcd(2a_1, 2b_1, 2c_1) \geq 2 > 1$ which is a contradiction.

Case 2: If $n = 2$, then we use congruence modulo 3 instead.

Observe that any integer $x \equiv 0, 1, 2 \pmod{3}$. Hence, any perfect square $x^2 \equiv 0^2, 1^2, 2^2 \pmod{3}$. i.e., $x^2 \equiv 0, 1 \pmod{3}$.

Going back to the original equation with $n = 2$, we have $2a^2 + 3b^2 = 4c^2$.

The left hand side is $\equiv 2 \cdot 0 + 0, 2 \cdot 1 + 0 \pmod{3} \equiv 0, 2 \pmod{3}$.

The right hand side is $\equiv 4 \cdot 0, 4 \cdot 1 \pmod{3} \equiv 0, 1 \pmod{3}$.

So, for the two sides to be the same, we must have both sides congruent to $0 \pmod{3}$.

Retracing the steps, this can happen when $a^2, c^2 \equiv 0 \pmod{3}$. i.e., $a, c \equiv 0 \pmod{3}$.

Say $a = 3a_1$ and $c = 3c_1$.

Then the original equation becomes $18a_1^2 + 3b^2 = 36c_1^2$ or $b^2 = 12c_1^2 - 6a_1^2 = 3(4c_1^2 - 2a_1^2)$.

The right hand side is divisible by 3 (i.e., $\equiv 0 \pmod{3}$).

Hence, we must have $b^2 \equiv 0 \pmod{3}$ which implies $b \equiv 0 \pmod{3}$. Say $b = 3b_1$.

But then $\gcd(a, b, c) = \gcd(3a_1, 3b_1, 3c_1) \geq 3 > 1$ which is again a contradiction.