

Putnam practice - Polynomials

Useful Facts on polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$:

1. Factor theorem:

If $P(r) = 0$, then $x - r$ divides $P(x)$.

More generally, if $P(r) = k$, then $P(x) = (x - r)Q(x) + k$ for some polynomials $Q(x)$.

Exercise 1: Suppose $P(x, y, z)$ is a polynomial in three variables. Prove that $(x - y)(y - z)(z - x)$ divides $Q(x, y, z) = P(x, y, z) + P(y, z, x) + P(z, x, y) - P(x, z, y) - P(y, x, z) - P(z, y, x)$.

2. Fundamental Theorem of Algebra and Vieta's formulas:

A degree n polynomial has n complex roots. Hence, one can factor

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - r_1)(x - r_2) \cdots (x - r_n).$$

Now, say $n = 4$. By expanding the product out, we get

$$a_4(-r_1 - r_2 - r_3 - r_4) = a_3 \quad \text{or} \quad r_1 + r_2 + r_3 + r_4 = -\frac{a_3}{a_4},$$

$$a_4(r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4) = a_2 \quad \text{or} \quad \sum_{1 \leq i < j \leq 4} r_i r_j = \frac{a_2}{a_4},$$

$$a_4(-r_1 r_2 r_3 - r_1 r_2 r_4 - r_1 r_3 r_4 - r_2 r_3 r_4) = a_1 \quad \text{or} \quad \sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = -\frac{a_1}{a_4},$$

$$a_4(r_1 r_2 r_3 r_4) = a_0 \quad \text{or} \quad r_1 r_2 r_3 r_4 = \frac{(-1)^4 a_0}{a_4}.$$

The above are Vieta's formulas and one can generalize them to degree n polynomials in a similar fashion.

Exercise 2: Factor the polynomial $x^4 - 6x^3 + 19x^2 - 32x + 30$ given that the sum of two of the roots is 4.

3. Binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

By putting in $x = y = 1$, one obtains the identity

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

By putting in $x = 1$ and $y = -1$, one obtains the identity

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

The binomial coefficients satisfy nice properties:

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}.$$

One can generalize the binomial theorem to any real power n : For $|x| < 1$,

$$(1+x)^n = \sum_{k=0}^{\infty} \underbrace{\frac{n(n-1)\cdots(n-k+1)}{k!}}_{\text{generalized } \binom{n}{k}} x^k$$

with $\binom{n}{k} = 1$ when $k = 0$, and $\binom{n}{k} = 0$ whenever n, k are integers satisfying $k < 0$ or $k > n$.

One can use it the approximate $\sqrt{17}$ as follows:

$$\sqrt{17} = (16+1)^{1/2} = 16^{1/2} \left(1 + \frac{1}{16}\right)^{1/2} = 4 \left(1 + \frac{1}{2} \cdot \frac{1}{16} + \frac{1}{2} \cdot \left(\frac{1}{2} - 1\right) \cdot \frac{1}{16^2} + \cdots\right) = 4 + \frac{1}{8} + \cdots$$

Exercise 3: Find a closed formula for the sum $\sum_{k=0}^n k \binom{n}{k}$

Exercise 4: Find a closed formula for the sum $\sum_{k=1}^n \binom{n}{k} \cos kx$.

Hint: Use Euler's identity $e^{i\theta} = \cos \theta + i \sin \theta$ and its consequences $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

Solutions:

Exercise 1:

First, let us treat x as a variable and y, z as constants. Then substituting $x = y$, we get $Q(y, y, z) = P(y, y, z) + P(y, z, y) + P(z, y, y) - P(y, z, y) - P(y, y, z) - P(z, y, y) = 0$. Hence, $x - y$ divides $Q(x, y, z)$.

Similarly, one can rotate and treat y as a variable with x, z as constants and z as a variable with x, y as constants. Then one can prove that $y - z$ and $z - x$ divide $Q(x, y, z)$ as well.

Exercise 2:

Suppose r_1, r_2, r_3, r_4 are the four roots of the polynomial.

By Vieta's formulas, we have $r_1 + r_2 + r_3 + r_4 = 6$ and $r_1 r_2 r_3 r_4 = 30$.

We know, without loss of generality, that $r_1 + r_2 = 4$. Then $r_3 + r_4 = 2$.

A trick here is to group $(x - r_1)(x - r_2)$ and $(x - r_3)(x - r_4)$ together. Then we obtain

$$x^4 - 6x^3 + 19x^2 - 32x + 30 = (x^2 - 4x + r_1 r_2)(x^2 - 2x + \frac{30}{r_1 r_2}).$$

Expanding and comparing the coefficient of x , we obtain

$$-2r_1 r_2 - \frac{120}{r_1 r_2} = -32 \quad \text{or} \quad (r_1 r_2)^2 - 16(r_1 r_2) + 60 = 0.$$

By factoring, one obtains $(r_1 r_2 - 6)(r_1 r_2 - 10) = 0$. Now, the coefficients of x^2 terms imply $19 = r_1 r_2 + 8 + \frac{30}{r_1 r_2}$, which forces $r_1 r_2 = 6$.

The quadratic $(x - r_1)(x - r_2)$ is $x^2 - 4x + 6$ which has roots $\frac{4 \pm \sqrt{-8}}{2} = 2 \pm \sqrt{2}i$.

The quadratic $(x - r_3)(x - r_4)$ is $x^2 - 2x + 5$ which has roots $\frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$.

Exercise 3:

From binomial theorem, $(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k$.

Differentiating both sides with respect to x , we obtain $n(1 + x)^{n-1} = \sum_{k=1}^n \binom{n}{k} k x^{k-1}$.

Putting in $x = 1$, we deduce that $n2^{n-1} = \sum_{k=0}^n k \binom{n}{k}$.

Exercise 4:

Let $S = \sum_{k=1}^n \binom{n}{k} \cos kx$.

By Euler's identity, we can deduce that

$$S = \sum_{k=1}^n \binom{n}{k} \frac{e^{ikx} + e^{-ikx}}{2} = \frac{1}{2} \left(\sum_{k=1}^n \binom{n}{k} e^{ikx} + \sum_{k=1}^n \binom{n}{k} e^{-ikx} \right).$$

By binomial theorem and taking care of $k = 0$ terms, we obtain

$$S = \frac{1}{2} \left((1 + e^{ix})^n + (1 + e^{-ix})^n - 2 \right) = \frac{1}{2} \left(e^{inx/2} (e^{-ix/2} + e^{ix/2})^n + e^{-inx/2} (e^{ix/2} + e^{-ix/2})^n - 2 \right)$$

after factoring out $e^{ix/2}$ and $e^{-ix/2}$ from the parenthese to make things more symmetric.

Applying Euler's identity, we obtain

$$S = \frac{1}{2} \left(e^{inx/2} \left(2 \cos \frac{x}{2} \right)^n + e^{-inx/2} \left(2 \cos \frac{x}{2} \right)^n - 2 \right) = \frac{1}{2} \left(\left(2 \cos \frac{x}{2} \right)^n \frac{e^{inx/2} + e^{-inx/2}}{2} - 1 \right).$$

Applying Euler's identity again, we obtain

$$S = \left(2 \cos \frac{x}{2} \right)^n \cdot \cos \frac{nx}{2} - 1.$$