Putnam practice - Probability solutions

1. A fair coin is flipped 100 times, what is the probability of getting an even number of heads?

Among the 100 flips, we can get either $0, 2, 4, \ldots, 98, 100$ heads.

Recall the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

By putting in $x = y = 1$, one obtains the identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

By putting in $x = 1$ and $y = -1$, one obtains the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Adding the above two equations together, we obtain

$$\sum_{k=0}^{n} \left(\binom{n}{k} + (-1)^k \binom{n}{k} \right) = \sum_{m \le n/2} 2 \binom{n}{2m} = 2^n.$$

Hence,

$$\sum_{m \le n/2} \binom{n}{2m} = 2^{n-1}.$$

Therefore, the required probability

$$= \frac{\binom{100}{0} + \binom{100}{2} + \cdots + \binom{100}{98} + \binom{100}{100}}{2^{100}} = \frac{2^{99}}{2^{100}} = \frac{1}{2}.$$

2. Three points are selected randomly on a unit circle. What is the probability that they will form an obtuse triangle?

Without loss of generality, one of the three points is at $(1, 0)$ on the x-axis.

Then, the other two points make angle $-\pi < \theta_1, \theta_2 \le \pi$ from the positive x-axis uniformly.

We can draw a square in a Cartesian plane with $-\pi < \theta_1, \theta_2 \le \pi$.

If $0 \le \theta_1, \theta_2 < \pi$ or $-\pi < \theta_1, \theta_2 \le 0$, then the triangle is obtuse since it is enclosed inside a semi-circle. (i.e., the first and third quadrants are okay.)

It remains to deal with the case that one angle is positive and the other angle is negative.

Suppose $\theta_1 > 0$ and $\theta_2 < 0$. If $\theta_1 - \theta_2 < \pi$, then the triangle is obtuse since it is enclosed inside a semi-circle. Otherwise, it is not obtuse. (i.e., the triangular region in the fourth quadrant bounded by $0 < \theta_1 \le \pi$, $-\pi < \theta_2 < 0$, $\theta_2 > \theta_1 - \pi$.)

We have a similar triangular region when $\theta_1 < 0$ and $\theta_2 > 0$.

Therefore, the probability

$$= \frac{\pi^2 + \pi^2 + \pi^2/2 + \pi^2/2}{(2\pi)^2} = \frac{3}{4}.$$

3. (2014) Suppose X is a random variable that takes on only non-negative integer values with expect values $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. Determine the smallest possible value of the probability of the event $X = 0$.

Let p_i be the probability that $X = i$ for $i = 0, 1, 2, 3, \ldots$.

This problem can be solved by means of **probability generating function** (by Tony Qiao):

$$F(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots = \sum_{i=0}^{\infty} p_i x^i \quad \text{with} \quad F(1) = \sum_{i=0}^{\infty} p_i = 1.$$

Note that $F(x)$ converges for $|x| \leq 1$ by comparison test with $\sum_{i=0}^{\infty} p_i = 1$. Then

$$F'(x) = \sum_{i=0}^{\infty} i p_i x^{i-1} \quad \text{with} \quad F'(1) = \sum_{i=0}^{\infty} i p_i = E[X] = 1.$$

Taking derivative again, we have

$$F''(x) = \sum_{i=0}^{\infty} i(i-1) p_i x^{i-2} \quad \text{with} \quad F''(1) = \sum_{i=0}^{\infty} i(i-1) p_i = E[X^2] - E[X] = 2 - 1 = 1.$$

Taking derivative one more time, we get

$$F'''(x) = \sum_{i=0}^{\infty} i(i-1)(i-2) p_i x^{i-3} \quad \text{with} \quad F'''(1) = \sum_{i=0}^{\infty} i(i-1)(i-2) p_i = E[X^3] - 3E[X^2] + 2E[X] = 1.$$

Now, by Taylor series of $F(x)$ with remainder term, for any $0 \leq x \leq 1$, there exists $c \in [x, 1]$ such that

$$F(x) = F(1) + \frac{F'(1)}{1!}(x-1) + \frac{F''(1)}{2!}(x-1)^2 + \frac{F'''(1)}{3!}(x-1)^3 + \frac{F^{(4)}(c)}{4!}(x-1)^4$$

Hence,

$$p_0 = F(0) = F(1) - \frac{F'(1)}{1!} + \frac{F''(1)}{2!} - \frac{F'''(1)}{3!} + \frac{F^{(4)}(c)}{4!}.$$

Since $F(x)$ has non-negative coefficients and $c \geq 0$, we must have $F^{(4)}(c) \geq 0$. Therefore,

$$p_0 \geq 1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

Note: Equality can be achieved by choosing $p_0 = \frac{1}{3}$, $p_1 = \frac{1}{2}$, $p_3 = \frac{1}{6}$, $p_n = 0$ for $n \geq 4$.

4. (2007) Let k be a positive integer. Suppose the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of integers

that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Among $1, 2, 3, \ldots, 3k+1$, there are $k+1$ integers $\equiv 1 \pmod 3$ (namely $3\cdot 0+1, 3\cdot 1+1, \ldots, 3\cdot k+1$), and there are k integers $\equiv 2 \pmod 3$ (namely $3 \cdot 0 + 2, 3 \cdot 1 + 2, \ldots, 3 \cdot (k-1) + 1$).

Since the sum of integers that have been written up to any time is not divisible by 3, we must start some integer $\equiv 1 \pmod 3$ or $\equiv 2 \pmod 3$.

Process 1: If the sum up to a certain time is $\equiv 1 \pmod 3$, the next integer in the list can be $\equiv 0, 1 \pmod 3$ but not $\equiv 2 \pmod 3$ since $1 + 0, 1 + 1 \not\equiv 0 \pmod 3$ but $1 + 2 \equiv 0 \pmod 3$.

Process 2: If the sum up to a certain time is $\equiv 2 \pmod 3$, the next integer in the list can be $\equiv 0, 2 \pmod 3$ but not $\equiv 1 \pmod 3$ since $2 + 0, 2 + 2 \not\equiv 0 \pmod 3$ but $2 + 1 \equiv 0 \pmod 3$.

Hence, if we start with an integer $\equiv 1 \pmod 3$, the next integer in the process that is not $\equiv 0 \pmod 3$ must be $\equiv 1 \pmod 3$. The sum would then be $\equiv 2 \pmod 3$.

Similarly, if we start with an integer $\equiv 2 \pmod 3$, the next integer in the process that is not $\equiv 0 \pmod 3$ must be $\equiv 2 \pmod 3$. The sum would then be $\equiv 1 \pmod 3$.

Consequently, by the above two starting positions together Processes 1 & 2, the random process of numbers with no sum at any time is not divisible by 3 must be

$$1, 0, \cdots, 0, 1, 0, \cdots, 0, 2, 0, \cdots, 0, 1, 0 \cdots, 0, 2, 0, \ldots, \cdots \pmod 3$$

or

$$2, 0, \cdots, 0, 2, 0, \cdots, 0, 1, 0, \cdots, 0, 2, 0 \cdots, 0, 1, 0, \ldots, \cdots \pmod 3.$$

Suppressing those integers $\equiv 0 \pmod 3$, we have

$$1, 1, 2, 1, 2, \cdots \pmod 3 \text{ or } 2, 2, 1, 2, 1, \cdots \pmod 3.$$

However, the second scenario cannot happen as the sequence starts with $2, 2 \pmod 3$ and then alternates between 1 and 2 afterwards. At any point of time, there are more integers $\equiv 2 \pmod 3$ than $\equiv 1 \pmod 3$ in the list. But this contradicts our observation at the beginning that there are more integers $\equiv 1 \pmod 3$ than $\equiv 2 \pmod 3$ among the integers $1, 2, 3, \ldots, 3k+1$.

Therefore, our list of random numbers (suppressing those divisible by 3) is of the form

$$1, 1, 2, 1, 2, \cdots, 1, 2 \pmod 3, \tag{1}$$

ending with $2 \pmod 3$ as there are 1 more integer $\equiv 1 \pmod 3$ than $\equiv 2 \pmod 3$.

There are $k+1$ choices for the first integer $1 \equiv \pmod 3$, k choices for the second integer $1 \equiv \pmod 3$, and so on. Similarly, there are k choices for the first integer $\equiv 2 \pmod 3$, $k-1$ choices for the second integer $\equiv 2 \pmod 3$, and so on.

Therefore, the total number of different arrangements of the $2k+1$ integers $\not\equiv 0 \pmod 3$ as in pattern (1) is $(k+1)! \cdot k!$.

Apart from the first slot, we are free to pick k slots from the remaining $3k$ slots and assign numbers $\equiv 0 \pmod 3$ into them. Then there are k multiples of 3 for the first such slot, $k-1$ multiplies of 3 for the second such slot, and so on. Afterwards, we can assign the numbers according to pattern (1) to the unpicked $2k+1$ slots.

Therefore, the probability is

$$\frac{\binom{3k}{k} \cdot k! \cdot (k+1)! \cdot k!}{(3k+1)!} = \frac{\frac{(3k)!}{k!(2k)!} \cdot k! \cdot (k+1)! \cdot k!}{(3k+1)!} = \frac{k!(k+1)!}{(2k)!(3k+1)}$$

as there are $(3k+1)!$ different ways to permute the $3k+1$ numbers.