Arithmetic with decimals: How to build on students' prior learning

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Decimal Numbers: Big Assumption

Teaching and learning of decimal numbers should take advantage of characteristics familiar to students.

- Whole number
- Fraction

Fractions and Whole/Decimal Numbers

- Fractions and decimal numbers are two notation systems for numbers that require units less than one.
- Numbers are expressed in terms of units.
  - Units for whole/decimal numbers: powers of 10
  - Units for fractions: unit fractions indicated by the denominator
    - $\frac{1}{D}$ = one of *D* equal partitioning of 1
- Decimal numbers have the characteristics of both fractions and whole numbers.
  - Decimal fractions: fractions with denominators of powers of 10
  - Extending decimal numeration system.

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Fractions and Decimal Numbers

Grade	Fractions	Decimal Numbers
1 & 2	Foundations – partitioning of shapes (1.G & 2.G)	
3	Formal introduction – focus on unit fractions (3.NF)	
4	Equivalent fractions +/−: like denominators ×: by whole numbers (4.NF)	Decimal numbers as “decimal notation” of fractions – 10th and 100th 4.NF.C Understand decimal notation for fractions, and compare decimal fractions. 4.NF.C.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. Footnote: … addition and subtraction with unlike denominators in general is not a requirement at this grade.

Fractions and Decimal Numbers

Grade	Fractions	Decimal Numbers
5	+/-: unlike denominators ×: by fractions ÷: whole number ÷ unit fraction unit fraction ÷ whole number (5.NF)	Decimal numbers through 1000th (5.NBT) +/−/×/÷: through 100th With concrete models, drawings, strategies based on place value, properties of operations; relate strategy to written method (5.NBT.B.7)
6	÷: fraction ÷ fraction Invert-and-multiply algorithm (6.NS)	Fluency with the standard algorithms (6.NS)

Structure of Whole/Decimal Numbers

- Positional: where a numeral is written matters.
- Each position (place) represents a specific value.

Positional vs. Non-positional System

5	五
55	五十五
505	五百五
5005	五千五
50005	五万五

Positional vs. Non-positional System

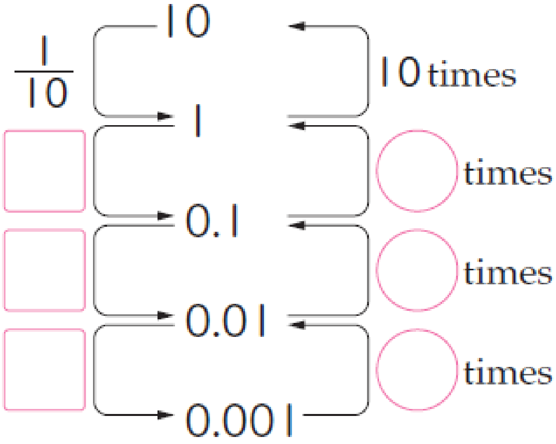
5	五
55	五十五
505	五百五
5005	五千五
50005	五万五

Whole/Decimal Numbers

- Positional: where a numeral is written matters.
- Each position (place) represents a specific value.
- Adjacent positions (places) are always in 1 to 10 relationship – 10 of a smaller units make up 1 of the next larger unit.

Place values

3 Write the number that goes in each of the □ on the right.
Also, write a number in each of the ○ .



Tokyo Shoseki (2010) Gr.4 p. A94

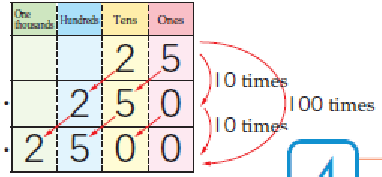
Multiplication/division by 10

3 If you multiply 25 10 times, then another 10 times, what number will you get?

1 How many times as much will it be if you multiply a number 10 times and then another 10 times?

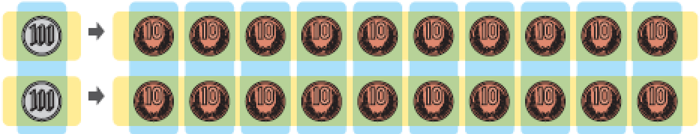
$25 \xrightarrow{10\text{ times}} 250 \xrightarrow{10\text{ times}} 2500$

☐ times



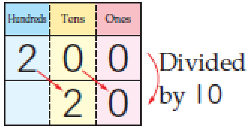
	One thousands	Hundreds	Tens	Ones
			2	5
25 × 10…		2	5	0
250 × 10…	2	5	0	0

4 What number is 200 divided by 10?



$200 \xrightarrow{\text{Divided by 10}} 20$

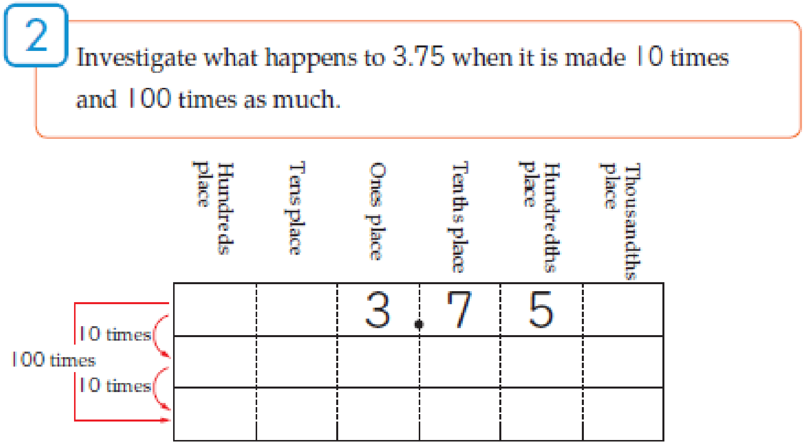
200 ÷ 10 = ☐



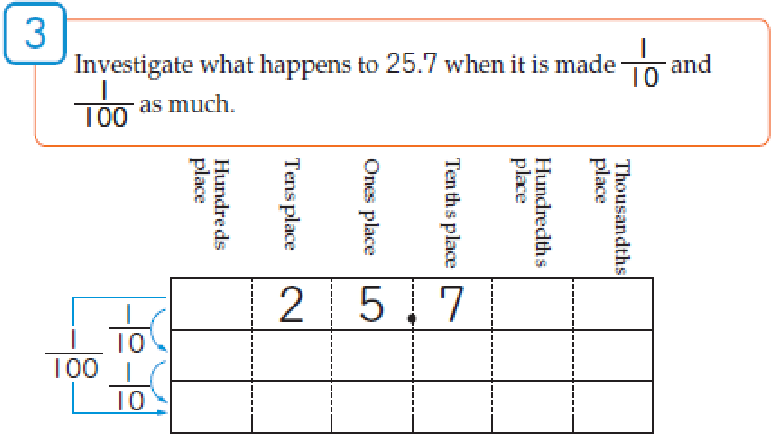
Hundreds	Tens	Ones
2	0	0
	2	0

Tokyo Shoseki (2010) Gr.3 p. A86

Multiplication/division by 10



2 Investigate what happens to 3.75 when it is made 10 times and 100 times as much.



3 Investigate what happens to 25.7 when it is made $\frac{1}{10}$ and $\frac{1}{100}$ as much.

Tokyo Shoseki (2010) Gr.5 pp. A8 & A9

Whole/Decimal Number
Relative Size of Numbers

- With the decimal numeration system, a number is represented as accumulation of units (powers of 10).
- 2345 is made of
    - 2 units of 1000
    - 3 units of 100
    - 4 units of 10
    - 5 units of 1

Whole/Decimal Number
Relative Size of Numbers

- With the decimal numeration system, a number is represented as accumulation of units (powers of 10).
- 23.45 is made of
    - 2 units of 10
    - 3 units of 1
    - 4 units of 0.1
    - 5 units of 0.01

Whole/Decimal Number
Relative Size of Numbers

- With the decimal numeration system, a number is represented as accumulation of units (powers of 10).
- 2345 is made of
  - 2 units of 1000
  - 345 units of 1
- 2345 is made of
  - 23 units of 100
  - 45 units of 1
- 2345 is made of
  - 234 units of 10
  - 5 units of 1
- etc.

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Whole/Decimal Number
Relative Size of Numbers

- With the decimal numeration system, a number is represented as accumulation of units (powers of 10).
- 23.45 is made of
  - 23 units of 1
  - 45 units of 0.01
- 23.45 is made of
  - 234 units of 0.1
  - 5 units of 0.01
- 23.45 is made of
  - 2 units of 10
  - 34 units of 0.1
  - 5 units of 0.01
- etc.

Whole/Decimal Number
Relative Size of Numbers

Distinguishing the questions:

- What is the digit in the _______ place of this number?
- How many ______ are in this number?

Example: 43.148

- What is the digit in the hundredths place?
- How many hundredths are in this number?

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Regularity in Repeated Reasoning

Addition and subtraction of decimal numbers

- Thinking in terms of units other than 1
- Numbers can be added/subtracted only if they are referring to the same unit.

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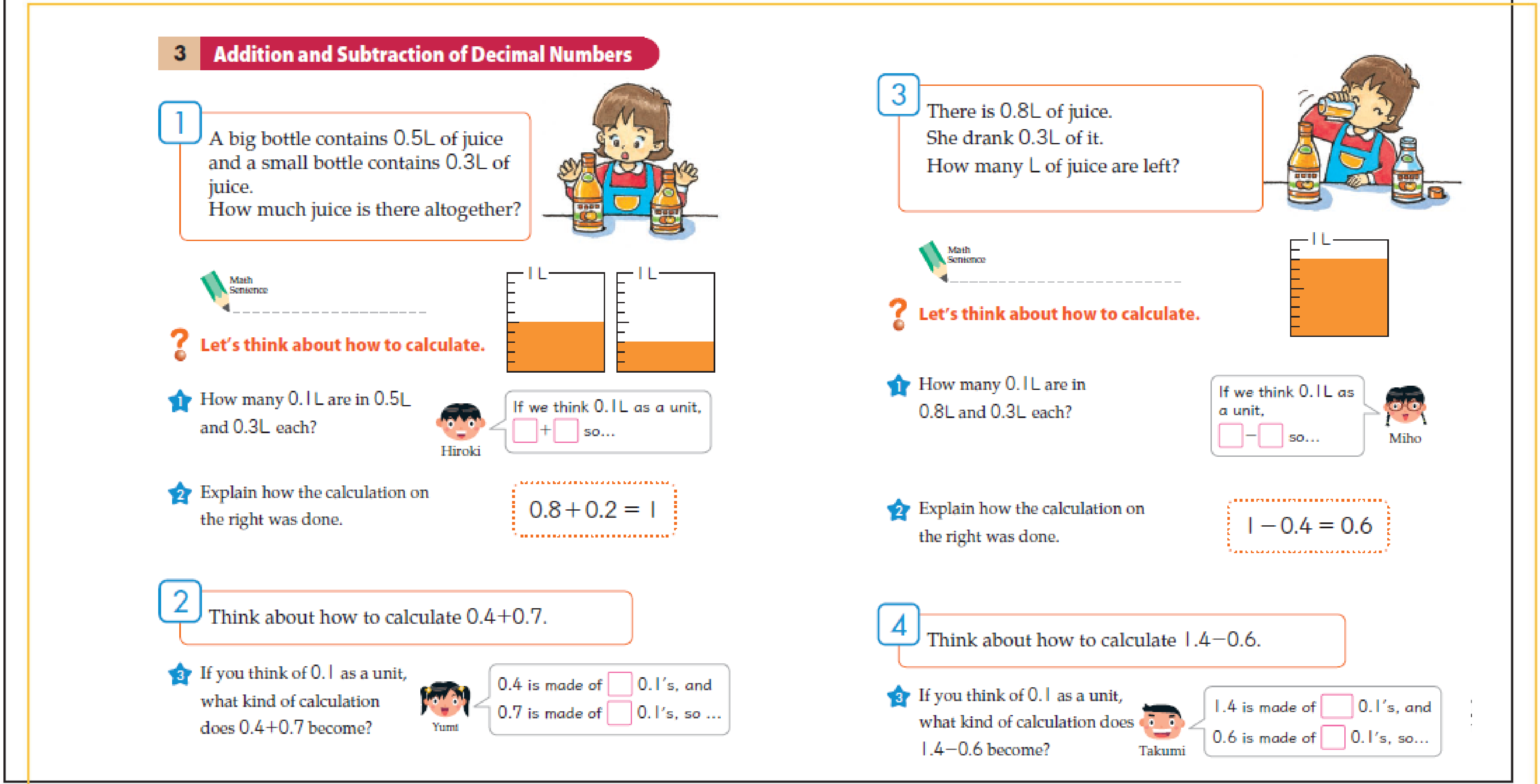


3 Addition and Subtraction of Decimal Numbers

1 A big bottle contains 0.5L of juice and a small bottle contains 0.3L of juice.
How much juice is there altogether?

Math Sentence ________________

? **Let's think about how to calculate.**

1 How many 0.1L are in 0.5L and 0.3L each?

Hiroki: If we think 0.1L as a unit, ☐+☐ so…

2 Explain how the calculation on the right was done.

0.8+0.2 = 1

2 Think about how to calculate 0.4+0.7.

3 If you think of 0.1 as a unit, what kind of calculation does 0.4+0.7 become?

Yumi: 0.4 is made of ☐ 0.1's, and 0.7 is made of ☐ 0.1's, so …

3 There is 0.8L of juice.
She drank 0.3L of it.
How many L of juice are left?

Math Sentence ________________

? **Let's think about how to calculate.**

1 How many 0.1L are in 0.8L and 0.3L each?

Miho: If we think 0.1L as a unit, ☐−☐ so…

2 Explain how the calculation on the right was done.

1 − 0.4 = 0.6

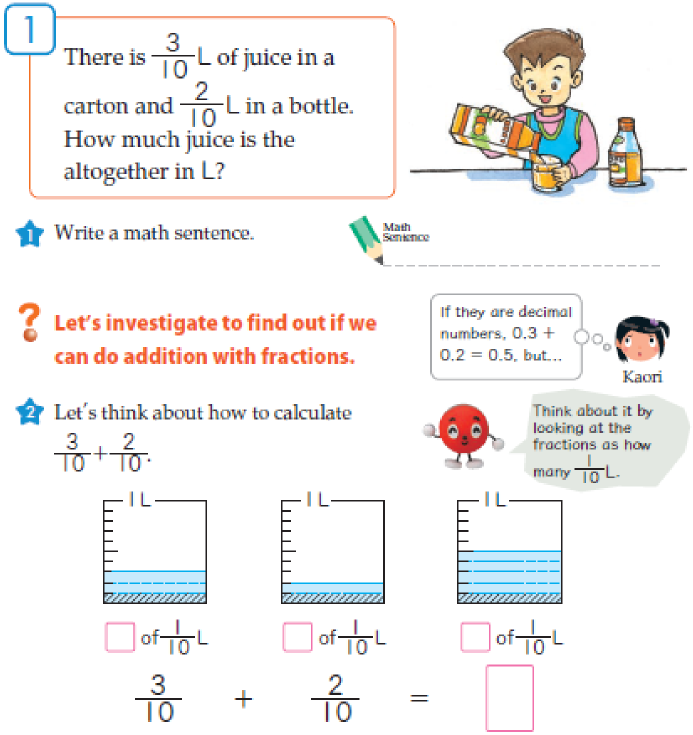
4 Think about how to calculate 1.4−0.6.

3 If you think of 0.1 as a unit, what kind of calculation does 1.4−0.6 become?

Takumi: 1.4 is made of ☐ 0.1's, and 0.6 is made of ☐ 0.1's, so…

Tokyo Shoseki (2010) Gr.3 pp. B19 & B20

Tokyo Shoseki (2010) Gr.3 pp. B51 & B52



1 There is $\frac{3}{10}$ L of juice in a carton and $\frac{2}{10}$ L in a bottle. How much juice is the altogether in L?

① Write a math sentence.

Math Sentence ______________________

? Let's investigate to find out if we can do addition with fractions.

If they are decimal numbers, 0.3 + 0.2 = 0.5, but…

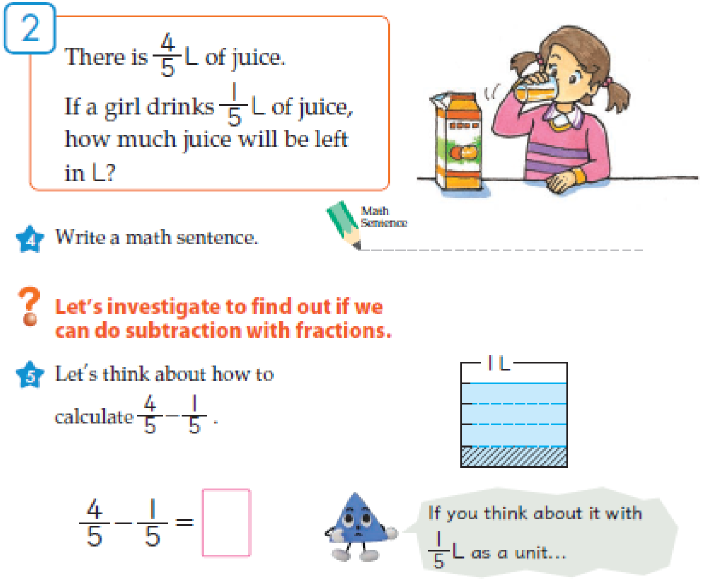
Kaori

② Let's think about how to calculate $\frac{3}{10}+\frac{2}{10}$.

Think about it by looking at the fractions as how many $\frac{1}{10}$ L.

1 L	1 L	1 L
☐ of $\frac{1}{10}$ L	☐ of $\frac{1}{10}$ L	☐ of $\frac{1}{10}$ L

$$\frac{3}{10} \quad + \quad \frac{2}{10} \quad = \quad \square$$



2 There is $\frac{4}{5}$ L of juice. If a girl drinks $\frac{1}{5}$ L of juice, how much juice will be left in L?

④ Write a math sentence.

Math Sentence ______________________

? Let's investigate to find out if we can do subtraction with fractions.

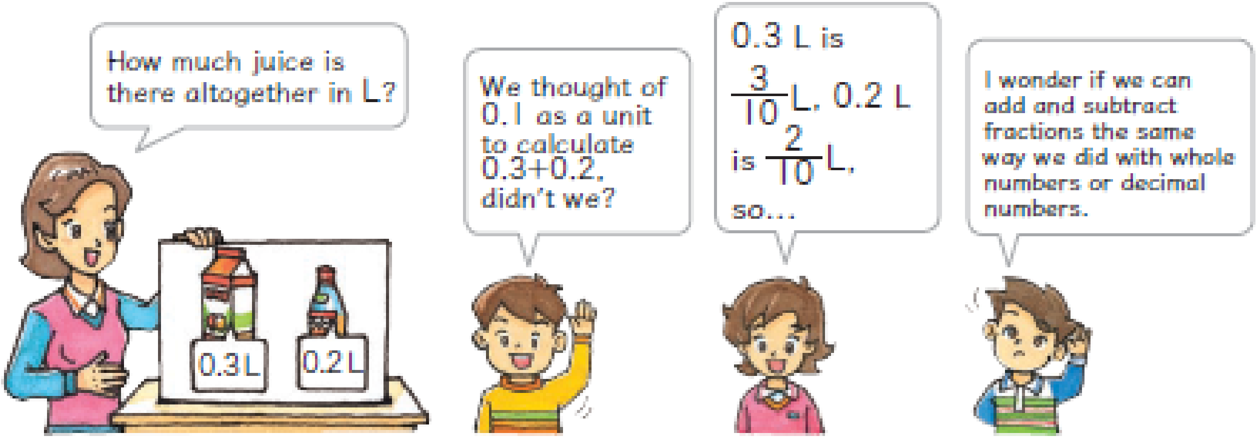
⑤ Let's think about how to calculate $\frac{4}{5}-\frac{1}{5}$.

1 L

$$\frac{4}{5}-\frac{1}{5}=\square$$

If you think about it with $\frac{1}{5}$ L as a unit…

Emphasizing repeated reasoning



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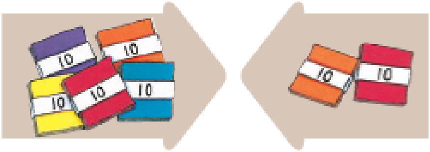
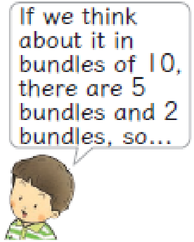
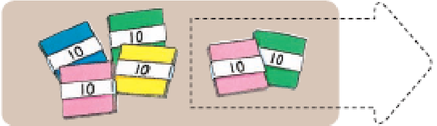
1 How many pieces of colored paper are there altogether?

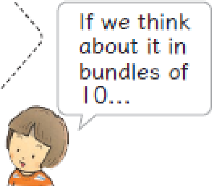




$50 + 20 = \square$

3 There are 60 pieces of colored paper.
If we use 20 pieces, how many will be left?



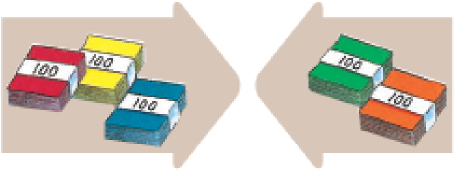


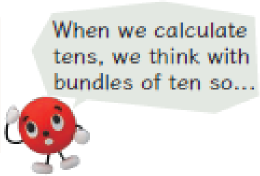
$60 - 20 = \square$

Tokyo Shoseki (2010) Gr.1 p. 128

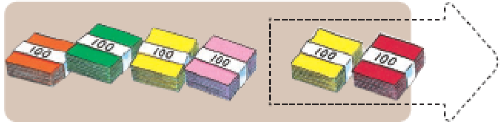
Tokyo Shoseki (2010) Gr.2 pp. A60 & A61

3 Think about how to calculate 300+200.





4 Think about how to calculate 600−200.



2 Explain the calculation by thinking the same way as $\boxed{3}$.

$600 - 200 = \square$

Multiplication of Decimal Numbers

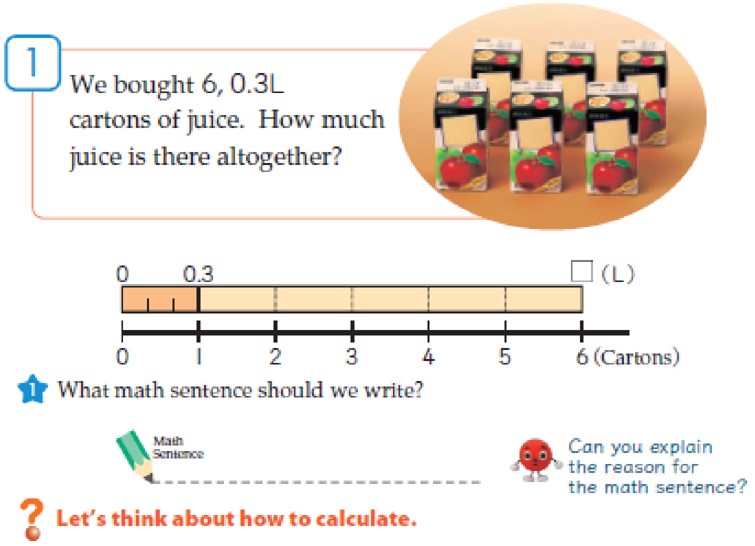
In the Japanese curriculum

- Multiplying decimal numbers by whole numbers in Grade 4
- Multiplying by decimal numbers in Grade 5

- CCSS 4.NF.4: multiplying fractions by whole numbers
- CCSS 5.NF.4: multiplying by fractions

Multiplying decimal numbers by whole numbers

- Continue to make use of decimal units



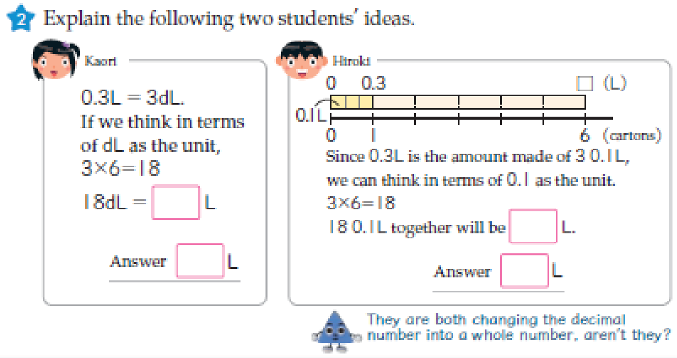
1 We bought 6, 0.3L cartons of juice. How much juice is there altogether?

1 What math sentence should we write?

Can you explain the reason for the math sentence?

? Let's think about how to calculate.

Tokyo Shoseki (2010) Gr.4
pp. B73 & B74



2 Explain the following two students' ideas.

Kaori

0.3L = 3dL.
If we think in terms of dL as the unit,
3×6=18
18dL = ☐ L

Answer ☐ L

Hiroki

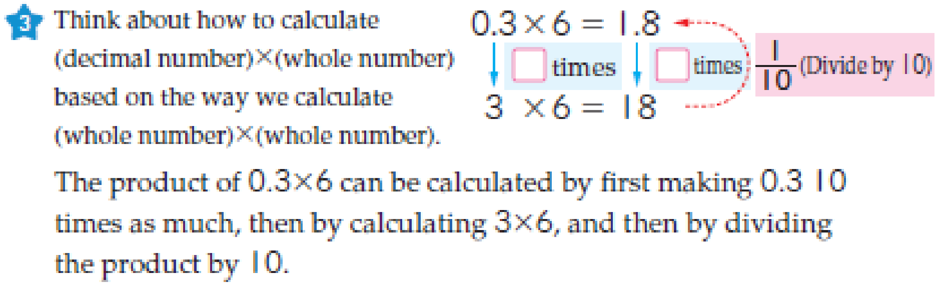
Since 0.3L is the amount made of 3 0.1L, we can think in terms of 0.1 as the unit.
3×6=18
18 0.1L together will be ☐ L.

Answer ☐ L

They are both changing the decimal number into a whole number, aren't they?

Multiplying decimal numbers by whole numbers

- Make use of property of multiplication



3 Think about how to calculate (decimal number)×(whole number) based on the way we calculate (whole number)×(whole number).

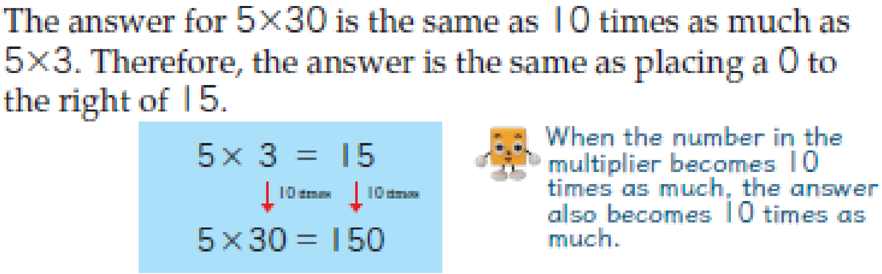
$0.3\times6=1.8$

↓ ☐ times ↓ ☐ times $\frac{1}{10}$ (Divide by 10)

$3\times6=18$

The product of 0.3×6 can be calculated by first making 0.3 10 times as much, then by calculating 3×6, and then by dividing the product by 10.

Tokyo Shoseki (2010) Gr.4 p. B74



The answer for 5×30 is the same as 10 times as much as 5×3. Therefore, the answer is the same as placing a 0 to the right of 15.

$5\times3=15$

↓ 10 times ↓ 10 times

$5\times30=150$

When the number in the multiplier becomes 10 times as much, the answer also becomes 10 times as much.

Tokyo Shoseki (2010) Gr.3 p. B64

Property of Multiplication

When a factor in a multiplication expression is multiplied by a number, the product will also by the same number times as much as the original product.

Example:

$$4 \times 5 = 20$$
$$(4 \times 3) \times 5 = 20 \times 3$$
$$4 \times (5 \times 2) = 20 \times 2$$

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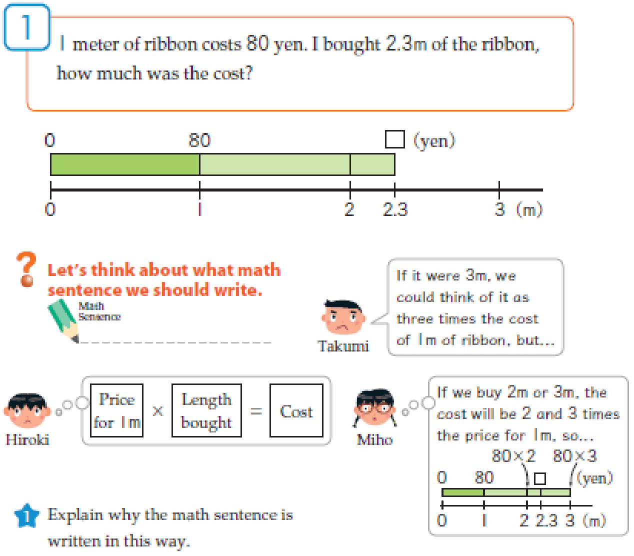
Multiplying by decimal numbers

- Making sense of multiplication by decimal numbers first

Tokyo Shoseki (2010) Gr.5 pp. A31 & A32



1 1 meter of ribbon costs 80 yen. I bought 2.3m of the ribbon, how much was the cost?

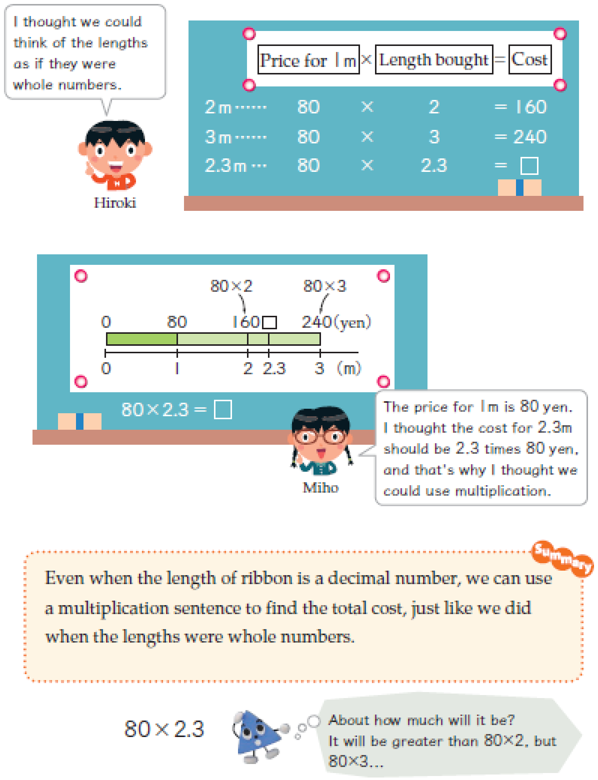
? Let's think about what math sentence we should write.

Takumi: If it were 3m, we could think of it as three times the cost of 1m of ribbon, but...

Hiroki: Price for 1m × Length bought = Cost

Miho: If we buy 2m or 3m, the cost will be 2 and 3 times the price for 1m, so...

1 Explain why the math sentence is written in this way.



Hiroki: I thought we could think of the lengths as if they were whole numbers.

Price for 1m × Length bought = Cost

2m…… 80 × 2 = 160

3m…… 80 × 3 = 240

2.3m… 80 × 2.3 = □

80×2.3 = □

Miho: The price for 1m is 80 yen. I thought the cost for 2.3m should be 2.3 times 80 yen, and that's why I thought we could use multiplication.

Summary

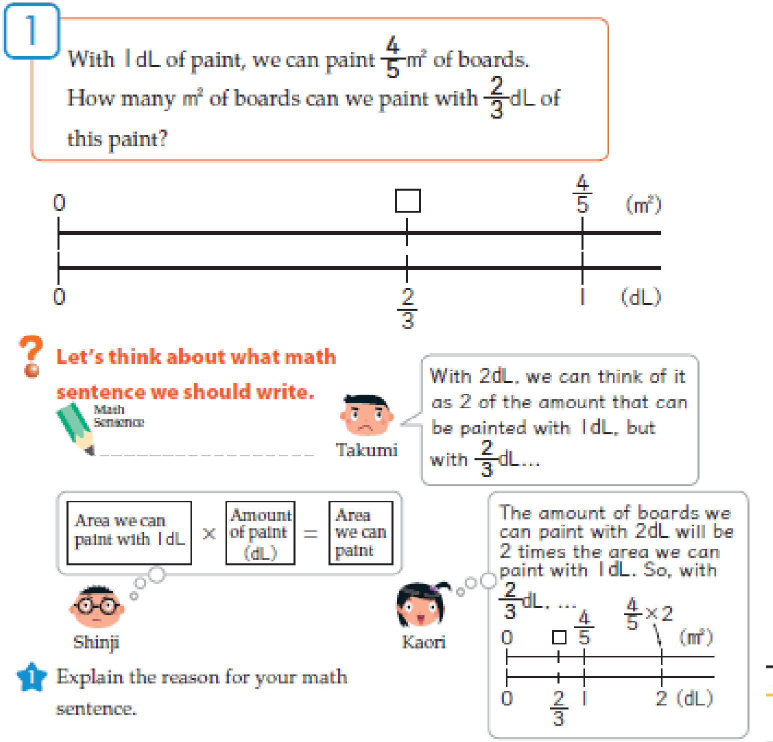
Even when the length of ribbon is a decimal number, we can use a multiplication sentence to find the total cost, just like we did when the lengths were whole numbers.

80×2.3

About how much will it be? It will be greater than 80×2, but 80×3...

Multiplying by fractions

- Making sense of multiplication by fractions first



1 With 1 dL of paint, we can paint $\frac{4}{5}$ m² of boards. How many m² of boards can we paint with $\frac{2}{3}$ dL of this paint?

Let's think about what math sentence we should write.

Math Sentence

Takumi: With 2dL, we can think of it as 2 of the amount that can be painted with 1dL, but with $\frac{2}{3}$ dL…

Area we can paint with 1dL × Amount of paint (dL) = Area we can paint

Shinji

Kaori: The amount of boards we can paint with 2dL will be 2 times the area we can paint with 1dL. So, with $\frac{2}{3}$ dL, …

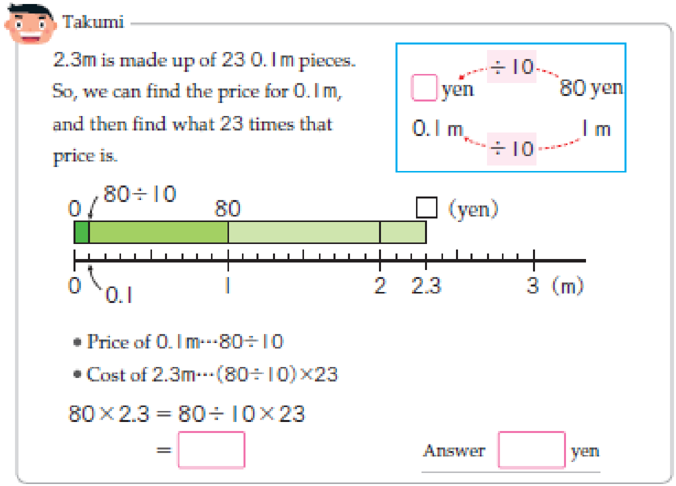
1 Explain the reason for your math sentence.

Tokyo Shoseki (2010) Gr.6 p. A34

Ways to multiply by decimal numbers

- Thinking in terms of decimal units
- Using property of multiplication

Tokyo Shoseki (2010) Gr.5 p. A33



Takumi

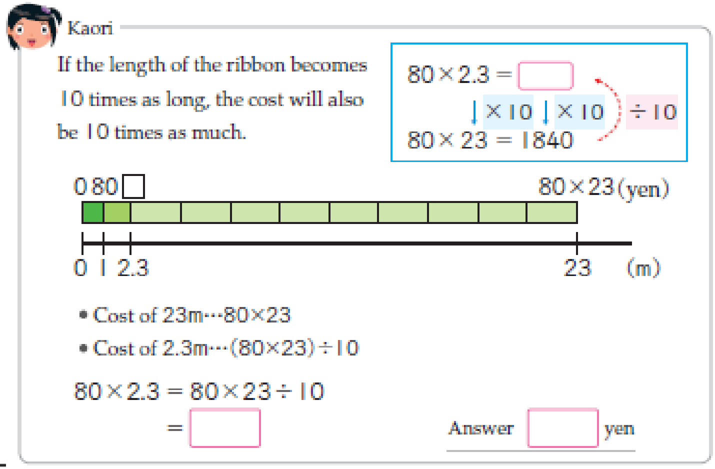
2.3m is made up of 23 0.1m pieces. So, we can find the price for 0.1m, and then find what 23 times that price is.

- Price of 0.1m···80÷10
- Cost of 2.3m···(80÷10)×23

$80\times2.3=80\div10\times23$

$=\square$

Answer ☐ yen



Kaori

If the length of the ribbon becomes 10 times as long, the cost will also be 10 times as much.

$80\times2.3=\square$

$\downarrow\times10\quad\downarrow\times10\qquad\div10$

$80\times23=1840$

- Cost of 23m···80×23
- Cost of 2.3m···(80×23)÷10

$80\times2.3=80\times23\div10$

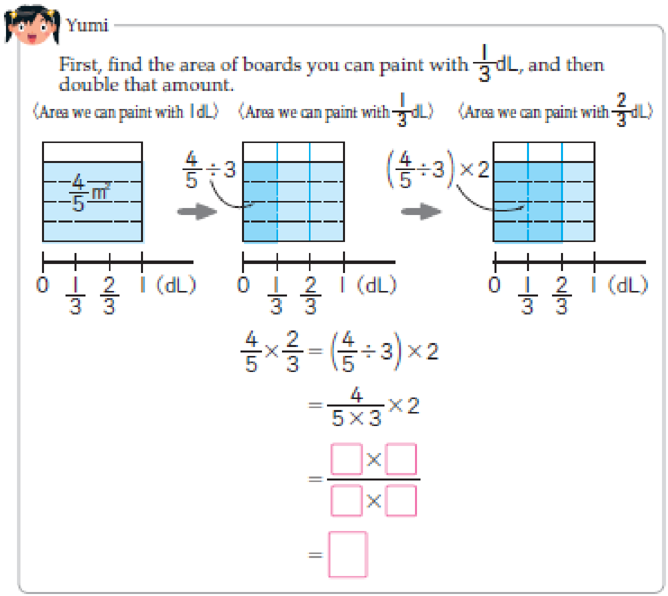
$=\square$

Answer ☐ yen

Ways to multiply by fractions

- Thinking in terms of fraction units
- Using property of multiplication

Tokyo Shoseki (2010) Gr.6 p. A25

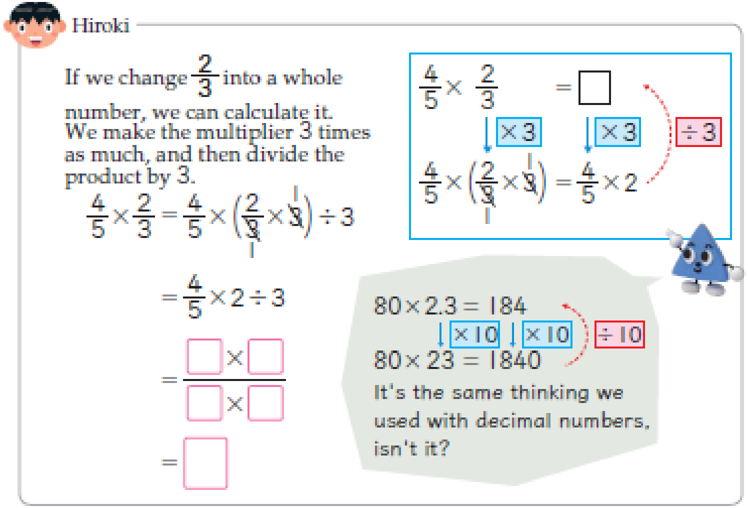


Yumi

First, find the area of boards you can paint with $\frac{1}{3}$dL, and then double that amount.

〈Area we can paint with 1dL〉 〈Area we can paint with $\frac{1}{3}$dL〉 〈Area we can paint with $\frac{2}{3}$dL〉

$$\frac{4}{5}\times\frac{2}{3}=\left(\frac{4}{5}\div 3\right)\times 2$$
$$=\frac{4}{5\times 3}\times 2$$
$$=\frac{\square\times\square}{\square\times\square}$$
$$=\square$$



Hiroki

If we change $\frac{2}{3}$ into a whole number, we can calculate it. We make the multiplier 3 times as much, and then divide the product by 3.

$$\frac{4}{5}\times\frac{2}{3}=\frac{4}{5}\times\left(\frac{2}{\underset{1}{\cancel{3}}}\times\overset{1}{\cancel{3}}\right)\div 3$$
$$=\frac{4}{5}\times 2\div 3$$
$$=\frac{\square\times\square}{\square\times\square}$$
$$=\square$$

$$\begin{tikzcd}\frac{4}{5}\times\frac{2}{3} \arrow[d, "\times 3"] & = & \square \\ \frac{4}{5}\times\left(\frac{2}{\underset{1}{\cancel{3}}}\times\overset{1}{\cancel{3}}\right) & = & \frac{4}{5}\times 2 \arrow[u, bend right, "\div 3"'] \arrow[from=1-3, to=2-3, phantom, "\times 3"]\end{tikzcd}$$

$$\begin{tikzcd}80\times 2.3 = 184 \arrow[d, "\times 10\ \times 10"] \\ 80\times 23 = 1840 \arrow[u, bend right, "\div 10"']\end{tikzcd}$$

It's the same thinking we used with decimal numbers, isn't it?

Ways to multiply by decimal numbers

- Thinking in terms of decimal units
- Using property of multiplication
- (If multiplication by fractions has been studied first) Change decimal numbers to fractions with powers of 10 as their denominators then multiply.

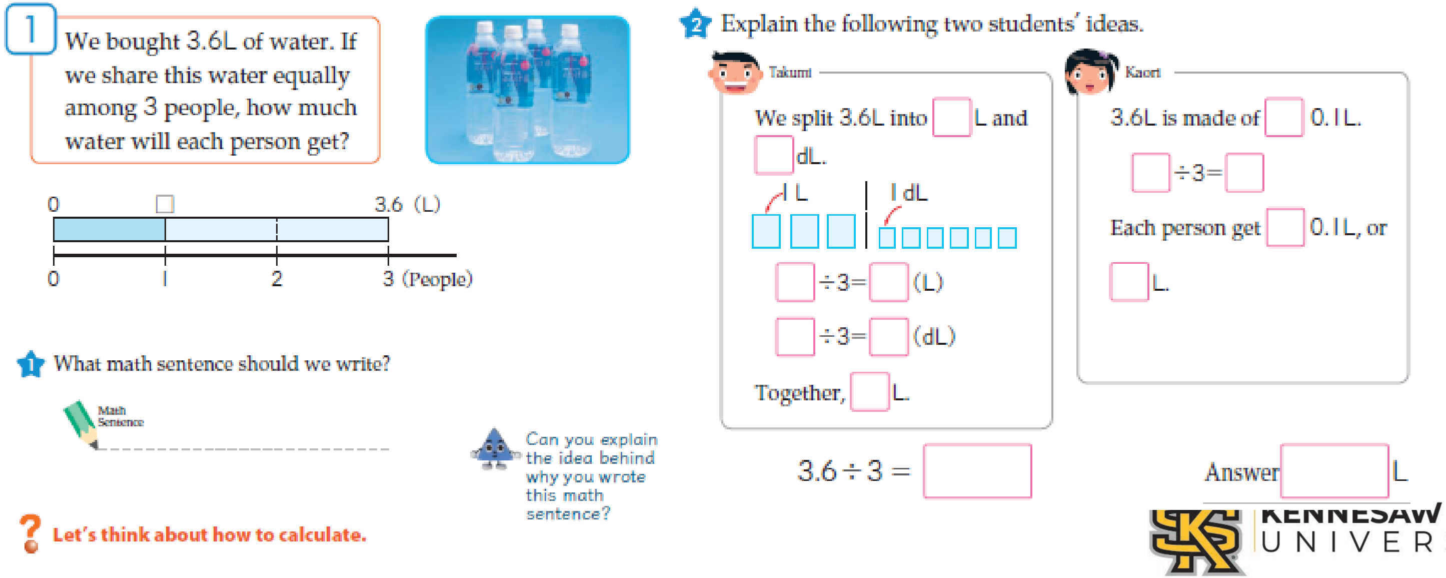
Division of decimal numbers

In the Japanese curriculum,

- Dividing decimal numbers by whole numbers in Grade 4
- Dividing by decimal numbers in Grade 5

Dividing decimal numbers by whole numbers

- Making use of context
- Making use of decimal units



1 We bought 3.6L of water. If we share this water equally among 3 people, how much water will each person get?

0 □ 3.6 (L)

0 1 2 3 (People)

1 What math sentence should we write?

Math Sentence

Can you explain the idea behind why you wrote this math sentence?

? **Let's think about how to calculate.**

2 Explain the following two students' ideas.

Takumi

We split 3.6L into □ L and □ dL.

1 L | 1 dL

□ ÷ 3 = □ (L)

□ ÷ 3 = □ (dL)

Together, □ L.

Kaori

3.6L is made of □ 0.1L.

□ ÷ 3 = □

Each person get □ 0.1L, or □ L.

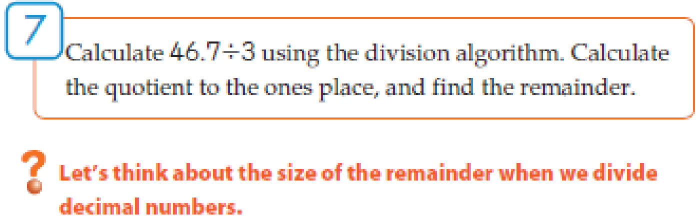
3.6 ÷ 3 = □

Answer □ L

Tokyo Shoseki (2010) Gr.4 pp. B80 & B81

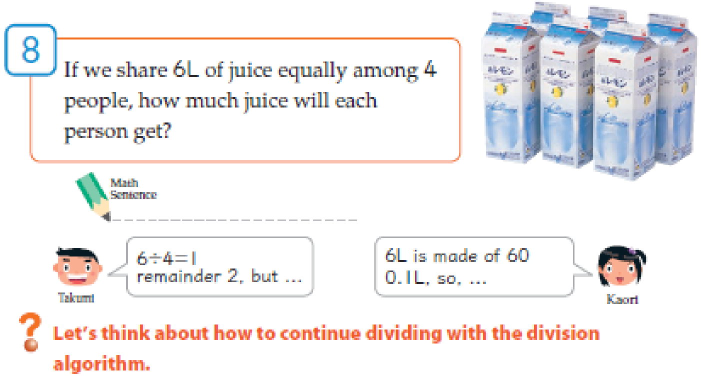
Dividing decimal numbers by whole numbers

- What if the dividend is not evenly divisible?
  - Remainder
  - Dividing on



7 Calculate 46.7÷3 using the division algorithm. Calculate the quotient to the ones place, and find the remainder.

? **Let's think about the size of the remainder when we divide decimal numbers.**



8 If we share 6L of juice equally among 4 people, how much juice will each person get?

Math Sentence

Takumi: 6÷4=1 remainder 2, but …

Kaori: 6L is made of 60 0.1L, so, …

? **Let's think about how to continue dividing with the division algorithm.**

Tokyo Shoseki (2010) Gr.4 pp. B86 & B87

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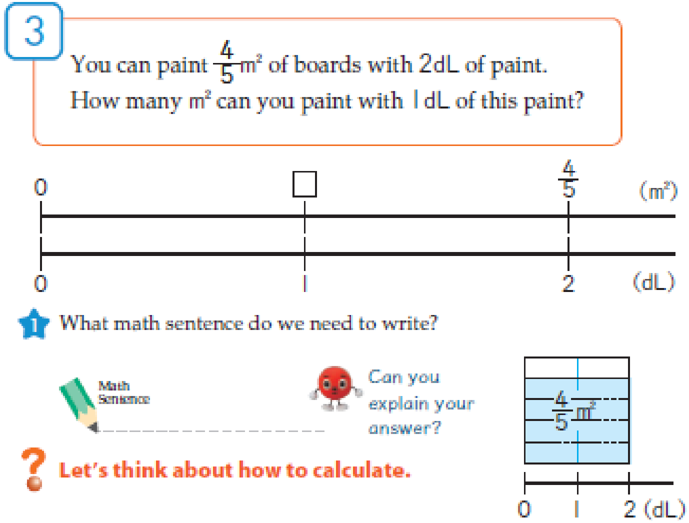
Dividing fractions by whole numbers

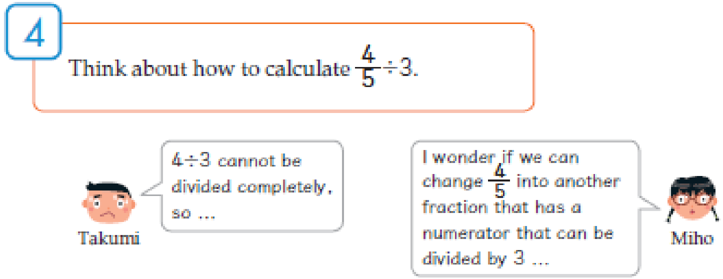


3 You can paint $\frac{4}{5}$m² of boards with 2dL of paint.
How many m² can you paint with 1dL of this paint?

1 What math sentence do we need to write?

Can you explain your answer?

? Let's think about how to calculate.



4 Think about how to calculate $\frac{4}{5} \div 3$.

Takumi: 4÷3 cannot be divided completely, so …

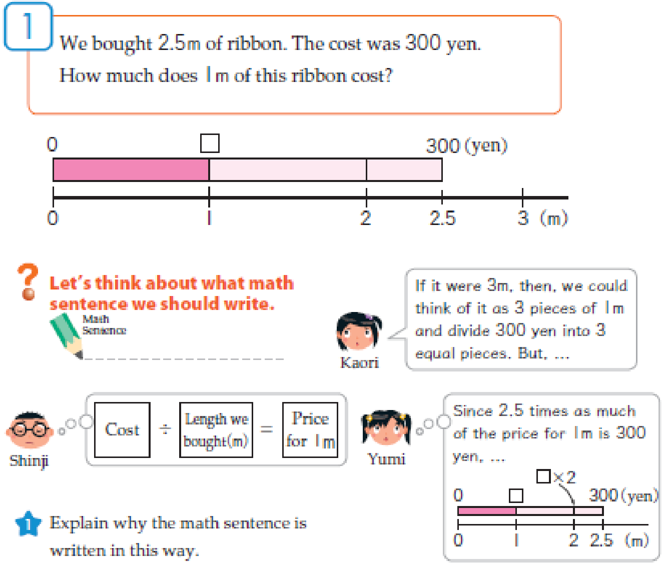
Miho: I wonder if we can change $\frac{4}{5}$ into another fraction that has a numerator that can be divided by 3 …

Tokyo Shoseki (2010) Gr.5 pp. B91 & B92

Dividing by decimal numbers

- Making sense of dividing by decimal numbers first

Tokyo Shoseki (2010) Gr.5 pp. A45 & A46



1 We bought 2.5m of ribbon. The cost was 300 yen. How much does 1m of this ribbon cost?

0 □ 300 (yen)

0 1 2 2.5 3 (m)

? **Let's think about what math sentence we should write.**

Math Sentence ____________________

Kaori: If it were 3m, then, we could think of it as 3 pieces of 1m and divide 300 yen into 3 equal pieces. But, ...

Shinji: Cost ÷ Length we bought(m) = Price for 1m

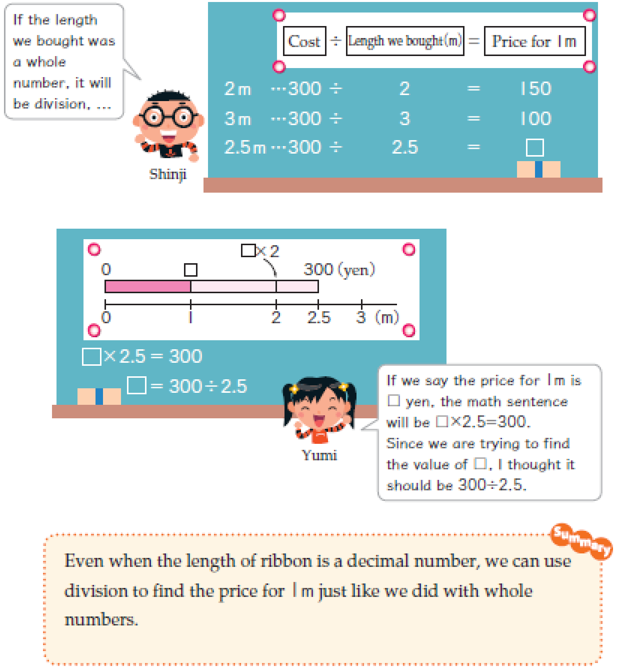
Yumi: Since 2.5 times as much of the price for 1m is 300 yen, ...

□×2

0 □ 300 (yen)

0 1 2 2.5 (m)

1 Explain why the math sentence is written in this way.



Shinji: If the length we bought was a whole number, it will be division, ...

Cost ÷ Length we bought(m) = Price for 1m

2m ···300 ÷ 2 = 150

3m ···300 ÷ 3 = 100

2.5m ···300 ÷ 2.5 = □

□×2

0 □ 300 (yen)

0 1 2 2.5 3 (m)

□×2.5 = 300

□ = 300÷2.5

Yumi: If we say the price for 1m is □ yen, the math sentence will be □×2.5=300. Since we are trying to find the value of □, I thought it should be 300÷2.5.

Summary

Even when the length of ribbon is a decimal number, we can use division to find the price for 1m just like we did with whole numbers.

Ways to divide by decimal numbers

- Thinking in terms of decimal units
- Using property of division

Property of Division

- If the dividend is multiplied by a number, the quotient will be the same number times as much as the original quotient.

Example:

$$48 \div 6 = 8$$
$$(48 \times 3) \div 6 = 8 \times 3$$

- If the dividend and the divisor are multiplied (or divided) by the same number, the quotient remains unchanged.

Example:

$$48 \div 6 = 8$$
$$(48 \times 3) \div (6 \times 3) = 8$$
$$(48 \div 3) \div (6 \div 3) = 8$$

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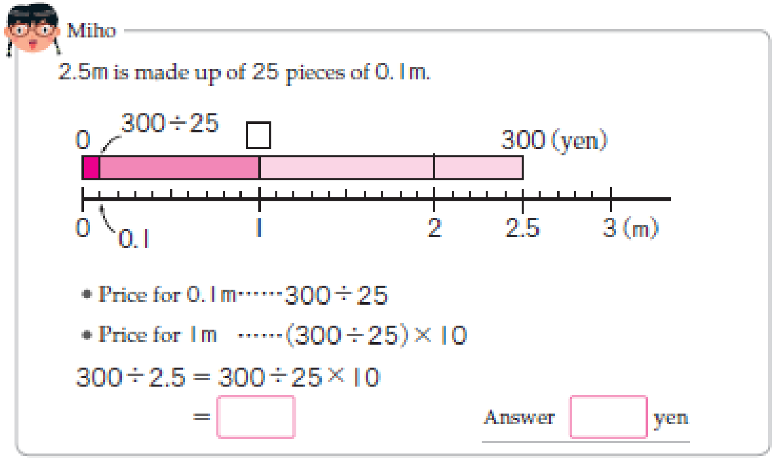
Ways to divide by decimal numbers

- Thinking in terms of decimal units
- Using property of division

Tokyo Shoseki (2010) Gr.5 p. A47



Miho

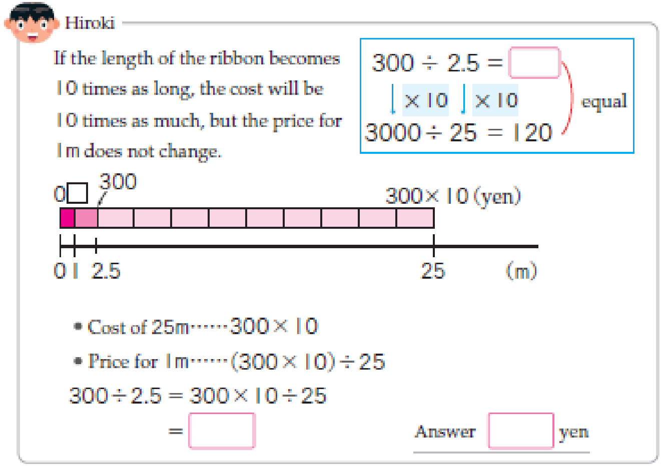
2.5m is made up of 25 pieces of 0.1m.

- Price for 0.1m……300÷25
- Price for 1m ……(300÷25)×10

300÷2.5 = 300÷25×10

= ☐

Answer ☐ yen



Hiroki

If the length of the ribbon becomes 10 times as long, the cost will be 10 times as much, but the price for 1m does not change.

300 ÷ 2.5 = ☐

↓×10 ↓×10 equal

3000 ÷ 25 = 120

- Cost of 25m……300×10
- Price for 1m……(300×10)÷25

300÷2.5 = 300×10÷25

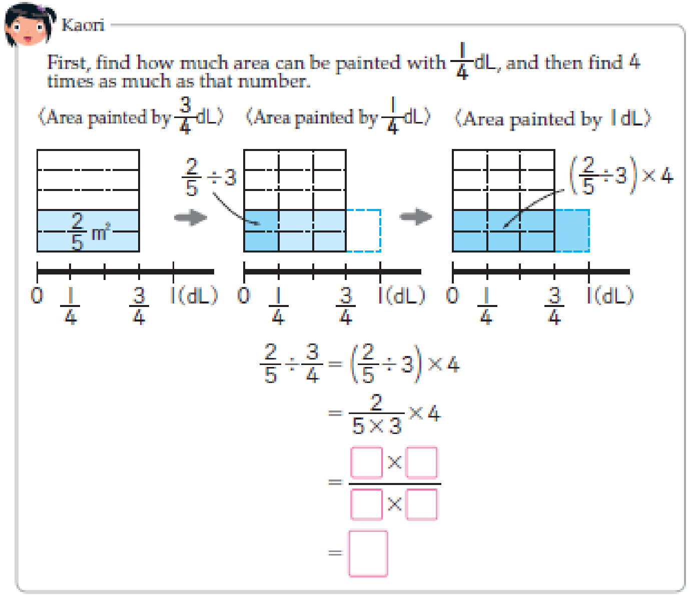
= ☐

Answer ☐ yen

Ways to divide by fractions

- Thinking in terms of fraction units
- Using property of division

Tokyo Shoseki (2010) Gr.5 p. A47



Kaori

First, find how much area can be painted with $\frac{1}{4}$dL, and then find 4 times as much as that number.

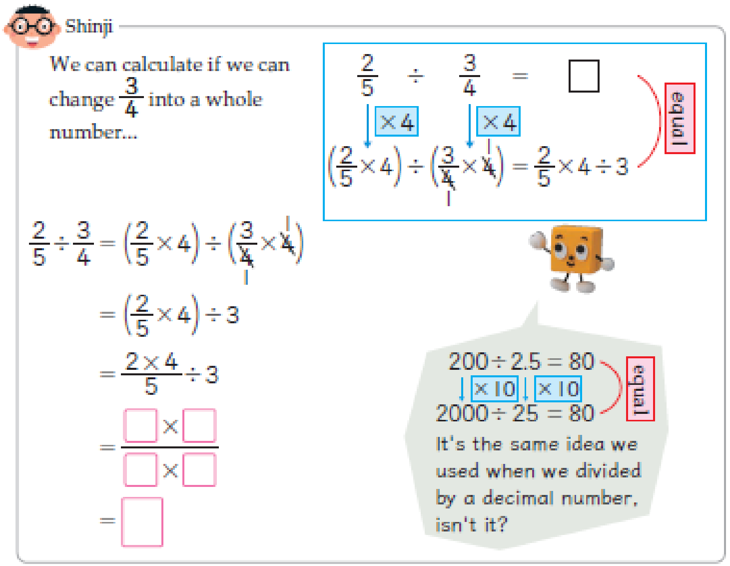
(Area painted by $\frac{3}{4}$dL) (Area painted by $\frac{1}{4}$dL) (Area painted by 1 dL)

$$\frac{2}{5}\div\frac{3}{4}=\left(\frac{2}{5}\div 3\right)\times 4$$

$$=\frac{2}{5\times 3}\times 4$$

$$=\frac{\square\times\square}{\square\times\square}$$

$$=\square$$



Shinji

We can calculate if we can change $\frac{3}{4}$ into a whole number...

$$\frac{2}{5}\div\frac{3}{4}=\square$$

$$\downarrow\boxed{\times 4}\qquad\downarrow\boxed{\times 4}$$

$$\left(\frac{2}{5}\times 4\right)\div\left(\frac{3}{\underset{1}{\cancel{4}}}\times\overset{1}{\cancel{4}}\right)=\frac{2}{5}\times 4\div 3$$

equal

$$\frac{2}{5}\div\frac{3}{4}=\left(\frac{2}{5}\times 4\right)\div\left(\frac{3}{\underset{1}{\cancel{4}}}\times\overset{1}{\cancel{4}}\right)$$

$$=\left(\frac{2}{5}\times 4\right)\div 3$$

$$=\frac{2\times 4}{5}\div 3$$

$$=\frac{\square\times\square}{\square\times\square}$$

$$=\square$$

$$200\div 2.5=80$$

$$\downarrow\boxed{\times 10}\ \downarrow\boxed{\times 10}$$

$$2000\div 25=80$$

equal

It's the same idea we used when we divided by a decimal number, isn't it?

Final Thoughts

- In order for students to look for and make use of structures in learning of decimal numbers, structures must become a focus in their learning of whole numbers (and fractions).
- It is helpful to have a curriculum flow that makes use of structures as a theme.

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Final Thoughts

- In order for students to look for and express regularity in repeated reasoning with decimal numbers, reasoning must become a focus in mathematics lessons.
- Tasks for lessons must be carefully chosen so that desired reasoning is more likely to arise from students.

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Thank you!

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