Ch 8.4: Trigonometric Form of Complex Numbers

In this section, we'll introduce

1. complex numbers and how to add or multiply two complex numbers
2. the complex plane having real and imaginary axis
3. complex numbers to trigonometric forms
4. powers of complex numbers (and look at DeMoivre's Theorem)
5. the n-th roots of complex numbers

Imaginary numbers

Notice that $x^2 = -1$ has no real solution. However, if we define

Definition

$$\boxed{i = \sqrt{-1}}$$

then either i or $-i$ satisfies the above equation. That is,

$$(\pm i)^2 = (\pm\sqrt{-1})^2 = -1$$

Example) Write the imaginary numbers in terms of i and simplify.

1. $\sqrt{-7}$ $= \sqrt{7 \cdot (-1)} = \sqrt{7}\,\sqrt{-1} = \sqrt{7}\,i$
2. $\sqrt{-81}$ $= \sqrt{9^2 \cdot (-1)} = \sqrt{9^2} \cdot \sqrt{-1} = 9i$
3. $-3\sqrt{-16}$ $= -3 \cdot \sqrt{4^2 \cdot (-1)} = -3\sqrt{4^2}\,\sqrt{-1} = -3 \cdot 4 \cdot i = -12i$

Complex numbers

Definition (Complex Numbers)

set of real #'s.

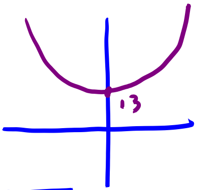
Complex numbers are of the form $a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$

$a=1 \quad b=6 \quad c=13$

Example) Find solutions to $x^2 + 6x + 13 = 0$. Simplify the solutions.

$ax^2 + bx + c = 0$

$f(x) = x^2 + 6x + 13$



real

No solution.

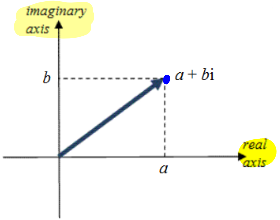
In Algebra
$$\begin{cases} 2 \text{ sols if } b^2 - 4ac > 0 \\ 1 \text{ sol if } b^2 - 4ac = 0 \\ \text{No real sol if } b^2 - 4ac < 0 \end{cases}$$

Since the solutions are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = \begin{cases} -3 + 2i \\ -3 - 2i \end{cases}$$

Complex plane

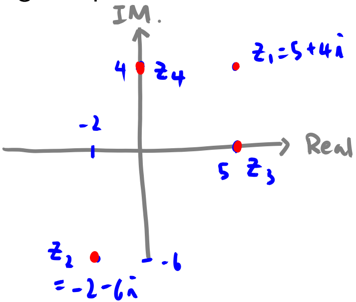
If we let the y-axis to represent the imaginary numbers, then



This is called the complex plane, $\mathbb{C}$.

Example) Graph the following complex numbers on the same complex plane.

1. $z_1 = 5 + 4i$
2. $z_2 = -2 - 6i$
3. $z_3 = 5$ $= 5 + 0 \cdot i$
4. $z_4 = 4i$ $= 0 + 4i$



Adding complex numbers

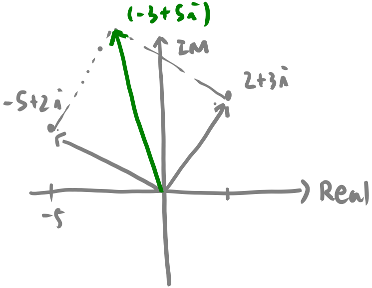
Adding two complex numbers can be done as follows: if $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, then

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Example)

1. $(2 + 3i) + (-5 + 2i)$ $= (2-5) + (3+2)i = (-3) + 5i$
2. $(-5 - 4i) + (-2 - \sqrt{2}i)$ $= (-5-2) + (-4-\sqrt{2})i$

   $= -3 + (-4-\sqrt{2})i$



Multiplying complex numbers

$i = \sqrt{-1} \Rightarrow i^2 = (\sqrt{-1})^2 = -1$

Noting that $i^2 = -1$, multiplying two complex numbers is as follows: if $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers,

$$z_1 z_2 = (a + bi)(c + di) = ac - bd + (ad + bc)i$$

Examples)

1. $(2 + 3i)(-5 + 2i)$
2. $\sqrt{-4}\sqrt{-9}$

$z_1 \cdot z_2 = (a+bi)(c+di)$

$= a \cdot c + a(di) + (bi)c + (bi)(di)$

$= ac + adi + bci + bd\underbrace{i^2}_{=-1}$

$= (ac - bd) + (ad+bc)i$

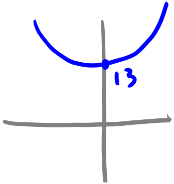
1. $-10 + 4i - 15i + 6\underbrace{i^2}_{=-1}$

$= -16 + (-11)i$

2. $\sqrt{4}\, i \sqrt{9}\, i = 2i\, 3i = 6i^2 = -6$

Example

Verify that $x = -3 + 2i$ is a solution to $x^2 + 6x + 13 = 0$.



$$\text{L.H.S} = (-3+2i)^2 + 6(-3+2i) + 13$$

$$= 9 + 2(-3)(2i) + (2i)^2 + (-18) + 12i + 13$$

$$= 9 - 12i - 4 - 18 + 12i + 13$$

$$= (9 - 4 - 18 + 13) + (-12i + 12i)$$

$$= 0 + 0i$$

$$= 0 = \text{R.H.S}$$

Example

Simplify

1. i^2
2. i^4
3. i^{22}
4. i^{57}

$$\left\{ i \xrightarrow[\times i]{} -1 \xrightarrow[\times i]{} -i \xrightarrow[\times i]{} 1 \right\} \xrightarrow[\times i]{} i \rightarrow -1 \rightarrow \dots$$

1. -1

2. 1

3. $$\begin{array}{r} 5 \\ 4 \overline{)22} \\ 20 \\ \hline 2 \end{array} \Rightarrow 22 = 4 \cdot 5 + 2 \Rightarrow i^{22} = i^{4\cdot 5+2} = i^{4\cdot 5} i^2$$
$$= (i^4)^5 i^2 = 1^5 \cdot i^2 = i^2$$

4. $$\begin{array}{r} 14 \\ 4 \overline{)57} \\ 46 \\ \hline 1 \end{array} \Rightarrow 57 = 4 \cdot 14 + 1 \Rightarrow i^{57} = i^{4\cdot 14} i^1 = (i^4)^{14} \cdot i = i^1$$

Example

Write each result in $a + bi$ form.

1. $\frac{2}{5-3i}$
2. $\frac{3-i}{2+i}$
3. $\frac{6+\sqrt{-36}}{3+\sqrt{-9}}$

1. $$\frac{2}{5-3i} \cdot \frac{5+3i}{5+3i} = \frac{10+6i}{25-(3i)^2} = \frac{10+6i}{25-(-9)} = \frac{10+6i}{34} = \frac{10}{34} + \frac{6}{34}i$$

$$= \frac{5}{17} + \frac{3}{17}i$$

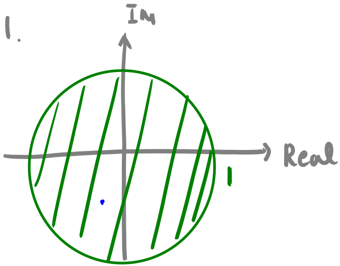
2. $$\frac{3-i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6-3i-2i+\overset{-1}{i^2}}{4-i^2} = \frac{5-5i}{5} = 1-i$$

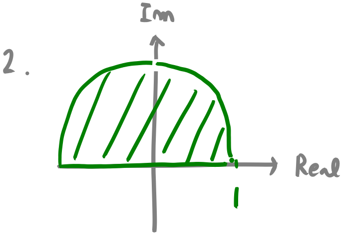
3. $$\frac{6+\sqrt{36}\,i}{3+\sqrt{9}\,i} = \frac{6+6i}{3+3i} = \frac{2(3+3i)}{3+3i} = 2.$$

Example

Graph the regions of the complex plane defined by the following:

1. $\{z \mid |z| \leq 1\}$
2. $\{z \mid |z| \leq 1 \;\;\&\;\; Im(z) \geq 0\}$





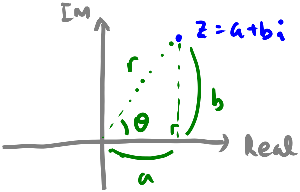
Complex numbers in Trig from

For any $z = a + bi$, and the corresponding angle θ,

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $r = \sqrt{a^2 + b^2}$

- $r = |z|$ is the magnitude (or modulus) of z.
- θ is called the argument of z and satisfies $\tan\theta = \frac{b}{a}$.



$z = a+bi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{a^2+b^2}$

$\cos\theta = \frac{a}{r} \Rightarrow a = r\cos\theta$

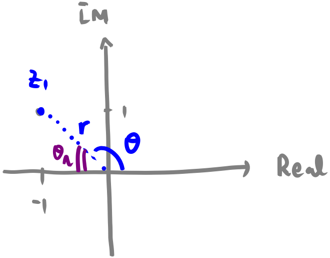
$\sin\theta = \frac{b}{r} \Rightarrow b = r\sin\theta$

Examples

State the quadrant of the complex numbers and convert each to trig form.

1. $z_1 = -1 + i$
2. $z_2 = 3 - 4i$

1.



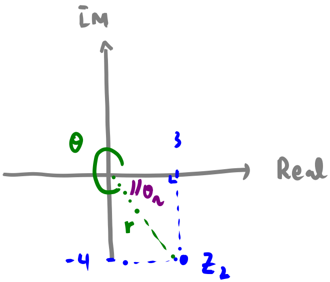
$r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\theta_r = \frac{\pi}{4} \Rightarrow \theta = \pi - \theta_r = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\Rightarrow z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

2. $z_2 = 3 - 4i$



$$r = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\tan\theta_r = \frac{4}{3} \Rightarrow \theta_r = \tan^{-1}\left(\frac{4}{3}\right)$$

$$= 53.13^\circ$$

$$\Rightarrow \quad \theta = 360^\circ - 53.13^\circ = 306.86^\circ$$

$$\Rightarrow z_2 = 3 - 4i$$

$$= r(\cos\theta + i\sin\theta)$$

$$= 5\left(\cos(306.86^\circ) + i\sin(306.86^\circ)\right)$$

From Trig to Rectangular (or Standard) form

Plot the following complex number, and write it in rectangular form.

1. $z = 2(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}))$
2. $z = 12(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))$

1. $z = 2\left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = \sqrt{3} + i$

2. $z = 12\left(\frac{\sqrt{3}}{2} + i \cdot \left(-\frac{1}{2}\right)\right) = 6\sqrt{3} - 6i$

Products and Quotient of Complex numbers in Trig Form

Let $z_1 = r_1(\cos\alpha + i\sin\alpha)$ and $z_2 = r_2(\cos\beta + i\sin\beta)$. Then,

$$\boxed{\begin{aligned} z_1 z_2 &= r_1 r_2[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\alpha-\beta) + i\sin(\alpha-\beta)], \;\; z_2 \neq 0 \end{aligned}}$$

Proof.

$$z_1 \cdot z_2 = r_1 r_2 (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= r_1 r_2 (\cos\alpha\cos\beta + (-1)\sin\alpha\sin\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta)$$

$$= r_1 r_2 (\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta))$$

$$= r_1 r_2 (\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$

Example

Let

$$z_1 = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Use the product formula to find

1. $z_1 z_2$
2. $\frac{z_1}{z_2}$

1. $z_1 \cdot z_2 = 3 \cdot 2 \left(\cos\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right) + i \sin\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)\right)$

$= 6 \left(\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)\right)$

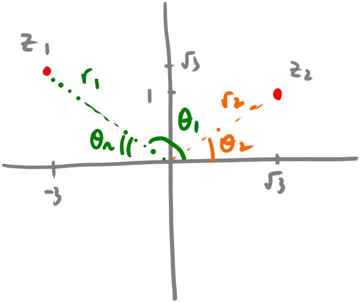
2. $\frac{z_1}{z_2} = \frac{3}{2} \left(\cos\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right) + i \sin\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right)\right)$

$= \frac{3}{2} \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right)$

Example

Let $z_1 = -3 + \sqrt{3}i$ and $z_2 = \sqrt{3} + i$.

1. Write z_1 and z_2 in trig form, and compute $z_1 z_2$.
2. Compute $\frac{z_1}{z_2}$.
3. Verify the product using the rectangular form.

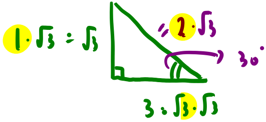


$$r_1 = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = \sqrt{4\cdot 3} = 2\sqrt{3}$$

$$\tan\theta = \frac{\sqrt{3}}{-3} \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}}{-3}\right) = \cancel{-30^\circ}$$

$$\Rightarrow \theta_1 = -30^\circ + 180^\circ = 150^\circ$$

OR



For z_2, $r_2 = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\tan\theta_2 = \frac{1}{\sqrt{3}} \Rightarrow \theta_2 = 30^\circ$

$$\Rightarrow \begin{cases} z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = 2\sqrt{3}(\cos(150^\circ) + i\sin(150^\circ)) \\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = 2(\cos(30^\circ) + i\sin(30^\circ)) \end{cases}$$

$$\Rightarrow \begin{cases} z_1 \cdot z_2 = r_1 \cdot r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) = 4\sqrt{3}(\underbrace{\cos 180^\circ}_{=-1} + i\underbrace{\sin 180^\circ}_{=0}) \\ \frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot (\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) = \sqrt{3}(\underbrace{\cos 120^\circ}_{=-\frac{1}{2}} + i\underbrace{\sin 120^\circ}_{=\frac{\sqrt{3}}{2}}) \end{cases}$$

3. verify.

$z_1 = -3 + \sqrt{3}i$ and $z_2 = \sqrt{3} + i$.

$$\Rightarrow z_1 \cdot z_2 = (-3+\sqrt{3}i)\cdot(\sqrt{3}+i) = -3\sqrt{3} - 3i + \underbrace{\sqrt{3}\cdot\sqrt{3}}_{=3}i + \sqrt{3}\,\underbrace{i^2}_{=-1}$$

$$= -4\sqrt{3} - 3i + 3i$$

$$= -4\sqrt{3} \quad (\checkmark)$$

$$\frac{z_1}{z_2} = \frac{-3+\sqrt{3}i}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{-3\sqrt{3} + 3i + 3i - \sqrt{3}\overbrace{i^2}^{=-1}}{(\sqrt{3})^2 - i^2} = \frac{-2\sqrt{3}+6i}{4}$$

$$= -\frac{\sqrt{3}}{2} + \frac{3}{2}i$$

DeMoivre's Theorem

Recall that if $z_1 = r_1(\cos\alpha + i\sin\alpha)$ and $z_2 = r_2(\cos\beta + i\sin\beta)$,

$$z_1 z_2 = r_1 r_2[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$$

Then, letting $z = z_1 = z_2$, one sees that $\left(r = r_1 = r_2 \,,\ \theta = \alpha = \beta \quad \text{i.e., } z = r(\cos\theta + i\sin\theta) \right)$

- $z^2 = r \cdot r\,[\cos(\theta+\theta) + i\sin(\theta+\theta)]$
  $= r^2\,[\cos(2\theta) + i\sin(2\theta)]$
- $z^3 = z^2 \cdot z = r^2 \cdot r\,[\cos(2\theta+\theta) + i\sin(2\theta+\theta)]$
  $= r^3\,[\cos(3\theta) + i\sin(3\theta)]$

Theorem (DeMoivre's Theorem)

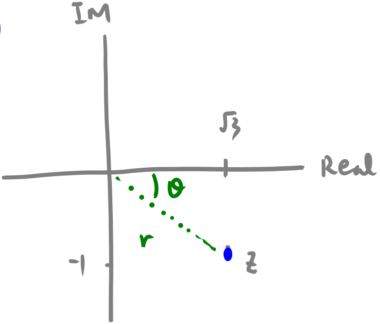
Given $z = r(\cos(\theta) + i\sin(\theta))$,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Examples

1. Given $z = \sqrt{3} - i$, compute z^7.
2. Given $z = -\frac{1}{2} - \frac{1}{2}i$, compute z^9.
3. Given $z = -1 - 5i$, compute z^5.

1)



$r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\tan(\theta) = \frac{-1}{\sqrt{3}} \Rightarrow \theta = -30^\circ$

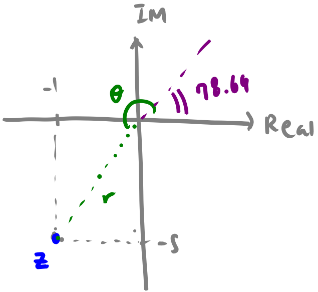
$\Rightarrow z = r(\cos\theta + i\sin\theta)$

$= 2(\cos(-30^\circ) + i\sin(-30^\circ))$

$\Rightarrow z^7 = 2^7(\cos(7\cdot(-30^\circ)) + i\sin(7\cdot(-30^\circ)))$

$= 128\left(-\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = -64\sqrt{3} + i\cdot 64$

3. Given $z = -\bar{1} - \bar{5}i$, compute z^5



$$r = \sqrt{(-1)^2 + (-5)^2} = \sqrt{1+25} = \sqrt{26}$$

$$\tan\theta = \frac{-5}{-1} = 5 \Rightarrow \theta = 78.69^\circ$$

using cal.

$$\Rightarrow \theta = 78.69^\circ + 180^\circ$$

$$= 258.69^\circ$$

$$\Rightarrow z = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{26}\left(\cos(\underline{258.69^\circ}) + i\sin(258.69^\circ)\right)$$

$$\Rightarrow z^5 = \sqrt{26}^5\left(\cos(5 \cdot 258.69^\circ) + i\sin(5 \cdot 258.69^\circ)\right)$$

$$= 26^{5/2}\left(-.834 + i(-.5512)\right)$$

The nth roots theorem

$= re^{i(\theta+2\pi k)}$

Let $z = r(\cos(\theta) + i\sin(\theta)) = r(\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k))$

$$\Rightarrow \quad z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

Alternatively, we can write

SAME

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\left(\frac{\theta+2\pi k}{n}\right)}.$$

$$z = r(\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k))$$

$$\Rightarrow z^n = r^n(\cos(n\theta + n\cdot 2\pi k) + i\sin(n\theta + n\cdot 2\pi k))$$

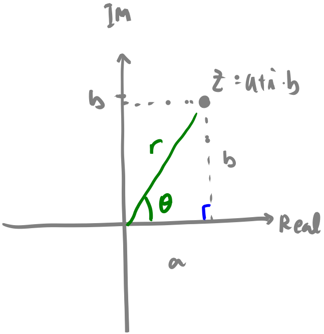
$$= r^n(\cos(n\theta) + i\sin(n\theta))$$

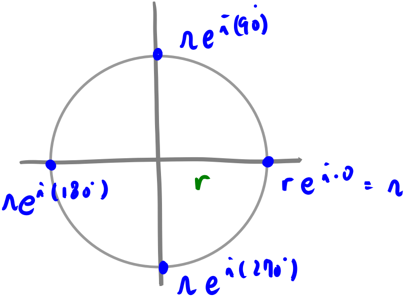
$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

$z = a + i \cdot b$ Rectangular

$= r(\cos\theta + i \cdot \sin\theta)$ Trig

$= r \cdot e^{i\theta}$ Euler

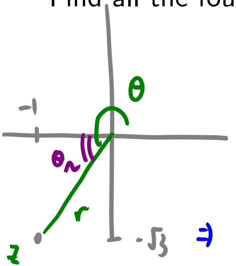




$$\begin{cases} \cos\theta = \dfrac{a}{r} \Rightarrow a = r\cos\theta \\ \sin\theta = \dfrac{b}{r} \Rightarrow b = r\sin\theta \end{cases}$$

Example

Find all the fourth roots of $-1 - i\sqrt{3}$.



$r = 2$

$z = 2(\cos 240^\circ + i \sin 240^\circ)$

$\theta = 180^\circ + \underbrace{60^\circ}_{= \theta_r} = 240^\circ$

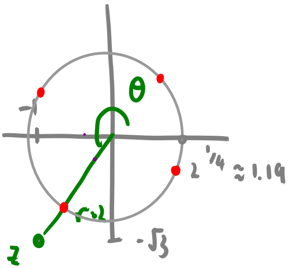
$$\Rightarrow z^{\frac{1}{4}} = 2^{\frac{1}{4}}\left(\cos\left(\frac{240^\circ}{4} + \frac{360^\circ \cdot k}{4}\right) + i \sin\left(\frac{240^\circ}{4} + \frac{360^\circ k}{4}\right)\right)$$

$$= 2^{\frac{1}{4}}\left[\cos(60^\circ + 90^\circ \cdot k) + i \sin(60^\circ + 90^\circ k)\right]$$

$$= \begin{cases} 2^{\frac{1}{4}}\left[\cos(60^\circ) + i \sin(60^\circ)\right] & k=0 \\ 2^{\frac{1}{4}}\left[\cos(150^\circ) + i \sin(150^\circ)\right] & k=1 \\ 2^{\frac{1}{4}}\left[\cos(240^\circ) + i \sin(240^\circ)\right] & k=2 \\ 2^{\frac{1}{4}}\left[\cos(330^\circ) + i \sin(330^\circ)\right] & k=3 \end{cases}$$

$$= \begin{cases} 2^{\frac{1}{4}} \left[\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \right] & k=0 \\ 2^{\frac{1}{4}} \left[-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2} \right] & k=1 \\ 2^{\frac{1}{4}} \left[-\frac{1}{2} + i \cdot \left(-\frac{\sqrt{3}}{2} \right) \right] & k=2 \\ 2^{\frac{1}{4}} \left[\frac{\sqrt{3}}{2} + i \left(-\frac{1}{2} \right) \right] & k=3 \end{cases}$$

Note: Given z, $z^{\frac{1}{n}}$ will have n answers.



Example

Solve the equation $x^5 - 1 = 0$. (That is, find all the fifth roots of 1) and graph their locations in the complex plane. Also, write the answers in rectangular form as well.

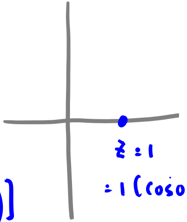
$$x^5 - 1 = 0 \Rightarrow x^5 = 1$$

$$= 1 \cdot [\cos(0^\circ + 360k) + i \sin(0^\circ + 360k)]$$

$$\Rightarrow x^{1/5} = 1^{1/5}\left[\cos\left(\frac{0+360k}{5}\right) + i \sin\left(\frac{0+360k}{5}\right)\right]$$

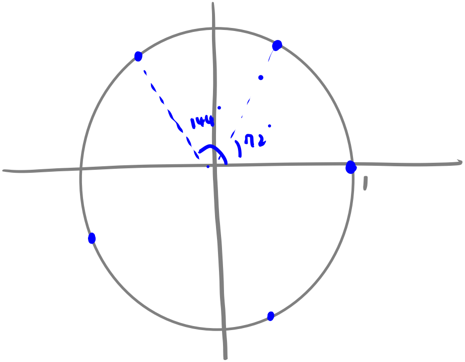
$$= 1 \cdot [\cos(72k) + i \sin(72k)]$$

$$= \begin{cases} 1 \cdot [\cos 0 + i \sin 0] & k=0 \\ 1 \cdot [\cos 72^\circ + i \sin 72] & k=1 \\ 1 \cdot [\cos 144^\circ + i \cdot \sin 144^\circ] & k=2 \\ 1 \cdot [\cos 216^\circ + i \sin 216^\circ] & k=3 \\ 1 \cdot [\cos 288^\circ + i \sin 288^\circ] & k=4 \end{cases}$$



$z = 1$

$= 1(\cos 0^\circ + i \sin 0)$



144
72
1

Homework for Ch 8.4

Show all work to get credit
1-15 odd, 17, 19, 20, 24, 29, 31, 34, 39, 41, 45, 47, 55, 56, 59-69 odd, 73