Normal and Binormal Vectors

Given a smooth curve $\mathbf{r}(t)$ and the corresponding unit tangent vector $\mathbf{T}(t)$, we define the unit normal vector $\mathbf{N}(t)$ as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Note that $\mathbf{N}(t)$ is perpendicular to $\mathbf{T}(t)$. (Why?)

$|T| = 1$

$T \cdot T = |T|^2 = 1$

$\Rightarrow (T \cdot T)' = (1)'$

$\Rightarrow T' \cdot T + T \cdot T' = 0$

$\Rightarrow 2T \cdot T' = 0$

$\Rightarrow T \cdot T' = 0$

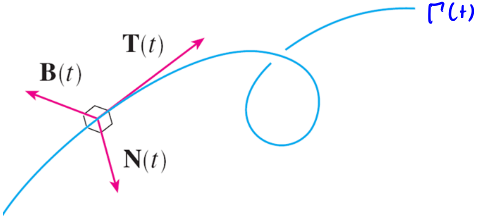
$\Rightarrow T \perp T'$

$\Rightarrow T \perp N$

And we define the binormal vector $\mathbf{B}(t)$ as

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

which is perpendicular to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is also a unit vector.



Example

Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

$$r'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 1\cdot k$$

$$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

$$\Rightarrow \boxed{T = \frac{r'}{|r'|} = \frac{\langle -\sin t, \cos t, 1\rangle}{\sqrt{2}}}$$

$$\Rightarrow T' = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle$$

$$|T'| = \frac{1}{\sqrt{2}}$$

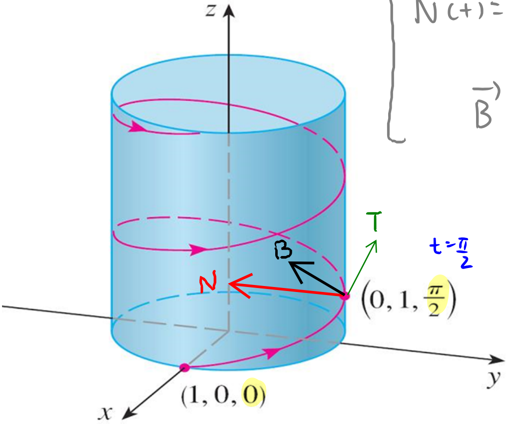
$$\Rightarrow \boxed{N(t) = \frac{T'}{|T'|} = \langle -\cos t, -\sin t, 0\rangle}$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{array}{ccc} + & - & + \\ \left|\begin{matrix} \hat{i} & \hat{j} & k \\ -\frac{1}{\sqrt{2}}\sin t & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{matrix}\right| \end{array}$$

$$= \hat{i} \begin{vmatrix} \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\sin t & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} -\frac{1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}} \\ -\cos t & 0 \end{vmatrix} + k \begin{vmatrix} -\frac{1}{\sqrt{2}}\sin t & \frac{\cos t}{\sqrt{2}} \\ -\cos t & -\sin t \end{vmatrix}$$

$$= \left\langle \frac{1}{\sqrt{2}}\sin t, \; -\frac{1}{\sqrt{2}}\cos t, \; \frac{1}{\sqrt{2}} \right\rangle$$

$$\Rightarrow \boxed{\vec{B} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle}$$



$$\begin{cases} T = \dfrac{r'}{|r'|} = \dfrac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}} \\ N(t) = \dfrac{T'}{|T'|} = \langle -\cos t, -\sin t, 0 \rangle \\ \vec{B} = \dfrac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle \end{cases}$$

$$T\left(\tfrac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \left\langle -\sin \tfrac{\pi}{2}, \cos \tfrac{\pi}{2}, 1 \right\rangle$$

$$= \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle$$

$$N\left(\tfrac{\pi}{2}\right) = \left\langle -\cos \tfrac{\pi}{2}, -\sin \tfrac{\pi}{2}, 0 \right\rangle$$

$$= \langle -0, -1, 0 \rangle$$

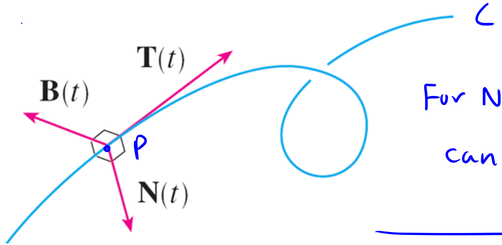
$$= \langle 0, -1, 0 \rangle$$

$$\vec{B}\left(\tfrac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \left\langle \sin \tfrac{\pi}{2}, -\cos \tfrac{\pi}{2}, 1 \right\rangle$$

$$= \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle$$

Normal and Osculating Plane

- The plane determined by the normal and binormal vectors **N** and **B** at a point P on a curve C is called the **normal plane** of C at P.
- The plane determined by the vectors **T** and **N** is called the **osculating plane** of C at P.



For Normal plane at P, can use $\vec{T}$ for a normal vector.

For osculating plane at P use $\vec{B}$ as its normal vector.

Example

Find the equations of the normal plane and osculating plane of the helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

at the point $P(0, 1, \frac{\pi}{2})$.

$T(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle -\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1 \rangle$

$= \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle$

$N(\frac{\pi}{2}) = \langle -\cos\frac{\pi}{2}, -\sin\frac{\pi}{2}, 0 \rangle$

$= \langle -0, -1, 0 \rangle$

$= \langle 0, -1, 0 \rangle$

$\vec{B}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle \sin\frac{\pi}{2}, -\cos\frac{\pi}{2}, 1 \rangle$

$= \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle$

$$\boxed{T = \frac{r'}{|r'|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}}$$

$$\boxed{N(t) = \frac{T'}{|T'|} = \langle -\cos t, -\sin t, 0 \rangle}$$

$$\boxed{\vec{B} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, \quad \rangle}$$

$\Rightarrow$ Eq. of the Normal plane at $P = (\overset{x_0}{0}, \overset{y_0}{1}, \overset{z_0}{\frac{\pi}{2}})$, using

$$\vec{r}\,'\left(\tfrac{\pi}{2}\right) = \langle \underset{a}{-1}, \underset{b}{0}, \underset{c}{1} \rangle \quad \left(\text{can use } T(t) = \tfrac{1}{\sqrt{2}}\langle -1, 0, 1\rangle\right)$$

$$\Rightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-(x - 0) + 0(y - 1) + (z - \tfrac{\pi}{2}) = 0$$

$$-x + z - \tfrac{\pi}{2} = 0$$

$$z - x = \tfrac{\pi}{2}.$$

Eq. of the Osculating plane at $P = (0, 1, \frac{\pi}{2})$ using $\langle \underset{a}{1}, \underset{b}{0}, \underset{c}{1} \rangle$ is

$$(x - 0) + 0(y - 1) + (z - \tfrac{\pi}{2}) = 0$$

$$x + z = \tfrac{\pi}{2}$$