The Lagrangian Equation
Developing a new perspective

- Hero of Alexandria (70 A.D.) - light reflections takes the shortest path
- Pierre de Fermat (1657) - light travels along a path that requires the least time
- Maupertuis (1747) - action is minimized through the “wisdom of God”
  - Action is a quantity with dimensions of energy x time
- William Hamilton (1834) –
  “Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.”
The Lagrangian

\[ L = T - U \]

- \( T \) = Kinetic Energy
- \( U \) = Potential Energy
- Hamilton’s principle asserts that for any given system there is an extremum (maximum or minimum) when the time integral is taken of the difference of Kinetic and Potential Energies.

\[ \int_{t_1}^{t_2} (T - U) dt = \text{Min or Max (Minimum in almost all dynamic systems)} \]

- Let \( S = \int_{t_1}^{t_2} L \, dt \) “Action”
Calculus of Variations

- Begun by Newton
- Developed by Johann Bernoulli, Jakob Bernoulli, and Leonhard Euler
- Important contributions made by Joseph Lagrange, Hamilton, and Jacobi

Leonhard Euler
(1707 - 1783)
Calculus of Variations

Consider a function for any generalized coordinate system

\[ x_a(t) = x_0(t) + \alpha \beta(t) \]

where \( x_0(t) \) produces a min value for \( S \), \( \alpha \) is a number, and \( \beta(t) \) is 0 at both end points of our interval.

- When this is integrated, the \( t \) is integrated out and \( S \) becomes a number dependent on \( \alpha, t_1, \) & \( t_2 \).

How does \( S \) depend on \( \alpha \)?

\[
\frac{\partial}{\partial \alpha} S[x_a(t)] = \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha} L \, dt
\]

\[
\frac{\partial}{\partial \alpha} S[x_a(t)] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x_a} \frac{\partial x_a}{\partial \alpha} + \frac{\partial L}{\partial \dot{x}_a} \frac{\partial \dot{x}_a}{\partial \alpha} \right) dt
\]

Use the Chain rule to separate
Calculus of Variations

Use substitution:

From our initial function:
\[ x_a(t) = x_0(t) + \alpha \beta(t) \]

\[ \frac{\partial x_a}{\partial \alpha} = \beta \quad \text{and} \quad \frac{\partial \dot{x}_a}{\partial \alpha} = \dot{\beta} \]

\[ \frac{\partial}{\partial \alpha} S[x_a(t)] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x_a} \frac{\partial x_a}{\partial \alpha} + \frac{\partial L}{\partial \dot{x}_a} \frac{\partial \dot{x}_a}{\partial \alpha} \right) dt \]

\[ \frac{\partial}{\partial \alpha} S[x_a(t)] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x_a} \beta + \frac{\partial L}{\partial \dot{x}_a} \dot{\beta} \right) dt \]
Calculus of Variations

Use integration by parts:

\[
\frac{\partial}{\partial \alpha} S[x_a(t)] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x_a} \beta + \frac{\partial L}{\partial \dot{x}_a} \dot{\beta} \right) dt
\]

Since \( x_0(t) \) produces a stationary value for \( S \),

\[
\frac{\partial}{\partial \alpha} S[x_a(t)] = 0
\]

Euler’s Equation:

\[
\frac{\partial L}{\partial x_0} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_0}
\]
A falling object

Consider a falling object (no motion in the x or z dimensions)

\[ T = \frac{1}{2} m \dot{y}^2 \quad U = mgy \quad L = \frac{1}{2} m \dot{y}^2 - mgy \]

Apply Euler’s Equation

\[ \frac{\partial L}{\partial y} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) \]

\[ -mg = \frac{d}{dt} (m\dot{y}) \]

\[ -mg = m\ddot{y} \]

\[ \ddot{y} = -g \]

The Lagrangian gives us Newton’s second law with respect to gravity.
A projectile

Consider a projectile launched in the positive x direction at some angle \( \theta \), where \( 0 < \theta < 90 \).

\[
\begin{align*}
T &= \frac{1}{2}m \dot{x}^2 + \frac{1}{2}m \dot{y}^2 \\
U &= mg y \\
L &= \frac{1}{2}m \dot{x}^2 + \frac{1}{2}m \dot{y}^2 - mg y
\end{align*}
\]

**X dimension**

\[
\frac{\partial L}{\partial x} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right)
\]

\[
0 = \frac{d}{dt} (m \dot{x})
\]

\[
0 = m \ddot{x}
\]

\[
\ddot{x} = 0
\]

**Y dimension**

\[
\frac{\partial L}{\partial y} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right)
\]

\[
-mg = \frac{d}{dt} (m \dot{y})
\]

\[
-mg = m \ddot{y}
\]

\[
\ddot{y} = -g
\]
An Orbiting body

Consider a body orbiting about another body with a central force acting on it.

**Position**
\[ x = r \cos(\theta) \quad y = r \sin(\theta) \]

**Velocity**
\[ \dot{x} = \dot{r} \cos(\theta) - r \dot{\theta} \sin(\theta) \quad \dot{y} = \dot{r} \sin(\theta) - r \dot{\theta} \cos(\theta) \]

**The Lagrangian**
\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \quad U = -\frac{GMm}{r} \]
\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{GMm}{r} \]
An Orbiting body

Consider a body orbiting about another body with a central force acting on it.

The Lagrangian

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{GMm}{r} \]

Apply Euler’s Equations

\[
\begin{align*}
\frac{\partial L}{\partial \dot{\theta}} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \\
0 &= \frac{d}{dt} (mr^2 \dot{\theta}) \\
mr^2 \dot{\theta} &= r \times P
\end{align*}
\]

\[
\begin{align*}
\frac{\partial L}{\partial r} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} \\
mr^2 \dot{\theta}^2 - \frac{GMm}{r^2} &= \frac{d}{dt} (mr \dot{r}) \\
mr \dot{\theta}^2 - \frac{GMm}{r^2} &= m \ddot{r}
\end{align*}
\]

\[
\ddot{r} = -\frac{GM}{r^2} + r \dot{\theta}^2
\]

\[ a_c = \frac{v^2}{r} = \frac{r^2 \dot{\theta}^2}{r} \]

Shows conservation of Angular momentum

Indicates the acceleration in the direction \( r \) is the central force acceleration + the tangential acceleration
Consider 3 charges interacting in a 2 dimensional plane

\[ T = \frac{1}{2} m \dot{x}_a^2 + \frac{1}{2} m \dot{y}_a^2 \]

\[ U = \frac{k q_a q_b}{\sqrt{(x_b-x_a)^2+(y_b-y_a)^2}} + \frac{k q_a q_c}{\sqrt{(x_c-x_a)^2+(y_c-y_a)^2}} \]

\[ L = \frac{1}{2} m \dot{x}_a^2 + \frac{1}{2} m \dot{y}_a^2 - \frac{k q_a q_b}{\sqrt{(x_b-x_a)^2+(y_b-y_a)^2}} - \frac{k q_a q_c}{\sqrt{(x_c-x_a)^2+(y_c-y_a)^2}} \]

**X dimension**

\[ \frac{\partial L}{\partial x_a} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_a} \]

\[- \frac{k q_a q_b (x_b-x_a)}{((x_b-x_a)^2+(y_b-y_a)^2)^{3/2}} - \frac{k q_a q_c (x_c-x_a)}{((x_c-x_a)^2+(y_c-y_a)^2)^{3/2}} = m \ddot{x}_a \]

\[- \frac{k q_a q_b}{(x_b-x_a)^2+(y_b-y_a)^2} \cos(\theta) - \frac{k q_a q_c}{(x_c-x_a)^2+(y_c-y_a)^2} \cos(\theta) = m \ddot{x}_a \]

**Y dimension**

\[ \frac{\partial L}{\partial y_a} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}_a} \]

\[- \frac{k q_a q_b (y_b-y_a)}{((x_b-x_a)^2+(y_b-y_a)^2)^{3/2}} - \frac{k q_a q_c (y_c-y_a)}{((x_c-x_a)^2+(y_c-y_a)^2)^{3/2}} = m \ddot{y}_a \]

\[- \frac{k q_a q_b}{(x_b-x_a)^2+(y_b-y_a)^2} \sin(\theta) - \frac{k q_a q_c}{(x_c-x_a)^2+(y_c-y_a)^2} \sin(\theta) = m \ddot{y}_a \]
A rotating pendulum

Consider a pendulum of length $l$ moving on a massless ring of radius $r$ and constant angular velocity $\omega$.

**Position**

$$x = r \cos(\omega t) + l \sin(\theta) \quad y = r \sin(\omega t) - l \cos(\theta)$$

**Velocity**

$$\dot{x} = -r \omega \sin(\omega t) + l \dot{\theta} \cos(\theta) \quad \dot{y} = r \omega \cos(\omega t) + l \dot{\theta} \sin(\theta)$$

**The Lagrangian**

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \quad U = mgy \quad L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$L = \frac{1}{2} m [r^2 \omega^2 + l^2 \dot{\theta}^2 + 2r \omega l \dot{\theta} \sin(\theta - \omega t)] - mg (r \sin(\omega t) - l \cos(\theta))$$
A rotating pendulum

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The Lagrangian

$$L = \frac{1}{2} m[r^2 \omega^2 + l^2 \dot{\theta}^2 + 2r \omega l \dot{\theta} \sin(\theta - \omega t)] - mg(r \sin(\omega t) - l \cos(\theta))$$

Apply Euler’s Equation

$$\frac{\partial L}{\partial \dot{\theta}} = m r \omega l \dot{\theta} \cos(\theta - \omega t) - m g l \sin(\theta)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \frac{d}{dt} \left[ m l^2 \ddot{\theta} + m r \omega l \dot{\theta} \cos(\theta - \omega t) \right]$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = m l^2 \ddot{\theta} + m r \omega l \dot{\theta} \cos(\theta - \omega t) - m r \omega^2 l \cos(\theta - \omega t)$$
A rotating pendulum

Consider a pendulum of length \( l \) moving on a massless ring of radius \( r \) and constant angular velocity \( \omega \).

*Apply Euler's Equation*

\[
\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}}
\]

\[
m r \omega^2 \cos(\theta - \omega t) - m g \sin(\theta) = m l^2 \ddot{\theta} + m r \omega^2 \cos(\theta - \omega t) - m r^2 \omega^2 \cos(\theta - \omega t)
\]

\[
ml^2 \ddot{\theta} = mr \omega^2 l \cos(\theta - \omega t) - m g \sin(\theta)
\]

Solve for \( \ddot{\theta} \)

\[
\ddot{\theta} = \frac{rw^2 \cos(\theta - \omega t)}{l} - \frac{gsin(\theta)}{l}
\]

Reduces to simple pendulum when \( \omega = 0 \).
Lagrangian Benefits

- Deals with derivatives of kinetic and potential energy differences which are invariant to coordinate transformations.
- Can greatly simplify complicated systems.
- Allows us to understand mechanical systems where it is difficult to describe all of the forces explicitly.
- Provides an alternative way to analyze a mechanical system: rather than seeing only cause and effect, we can now analyze a dynamical system by considering action minimization.
Sources

- https://en.wikipedia.org/wiki/Leonhard_Euler (photo)
- http://physicsinsights.org/rotating_polar_1.html